Spectra of Bochner-Riesz means on $L^p$

Yang Chen
College of Mathematics and Computer Science
Hunan Normal University
Hunan 410081, P. R. China
E-mail: yang_chen0917@163.com

Qiquan Fang
School of Science
Zhejiang University of Science and Technology
Hangzhou, Zhejiang 310023, P. R. China
E-mail: fendui@yahoo.com

Qiyu Sun
Department of Mathematics
University of Central Florida
Orlando, FL 32816, USA
E-mail: qiyu.sun@ucf.edu

November 10, 2015

Abstract
The Bochner-Riesz means are shown to have either the unit interval $[0, 1]$ or the whole complex plane as their spectra on $L^p, 1 \leq p < \infty$

1 Introduction and Main Results
Define Fourier transform $\hat{f}$ of an integrable function $f$ by
$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx$$
and extend its definition to all tempered distributions as usual. Consider Bochner-Riesz means \( B_\delta, \delta > 0 \), on \( \mathbb{R}^d \),

\[
\hat{B_\delta f}(\xi) := (1 - |\xi|^2)^\delta_+ \hat{f}(\xi),
\]

where \( t_+ := \max(t, 0) \) for \( t \in \mathbb{R} \) [2, 11, 20]. A famous conjecture in Fourier analysis is that the Bochner-Riesz mean \( B_\delta \) is bounded on \( L^p := L^p(\mathbb{R}^d) \), the space of all \( p \)-integrable functions on \( \mathbb{R}^d \) with its norm denoted by \( \| \cdot \|_p \), if and only if

\[
\delta > \left( d \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{1}{2} \right)_+, 1 \leq p < \infty. \tag{1.1}
\]

The requirement (1.1) on the index \( \delta \) is necessary for \( L^p \) boundedness of the Bochner-Riesz mean \( B_\delta \) [12]. The sufficiency is completely solved only for dimension two [5] and it is still open for high dimensions, see [3, 14, 21, 27, 28, 29, 31] and references therein for recent advances.

Denote the identity operator by \( I \). For \( \lambda \notin [0, 1] \), we first show that Bochner-Reisz means \( B_\delta \) is bounded on \( L^p \) if and only if its resolvents \( (zI - B_\delta)^{-1} \), which are multiplier operators with symbols \( (z - (1 - |\xi|^2)^\delta_+)^{-1} \), are bounded on \( L^p \) for all \( z \in \mathbb{C} \setminus [0, 1] \).

**Theorem 1.1.** Let \( \delta > 0 \) and \( 1 \leq p < \infty \). Then the following statements are equivalent to each other.

(i) The Bochner-Riesz mean \( B_\delta \) is bounded on \( L^p \).

(ii) \( (zI - B_\delta)^{-1} \) is bounded on \( L^p \) for all \( z \in \mathbb{C} \setminus [0, 1] \).

(iii) \( (z_0I - B_\delta)^{-1} \) is bounded on \( L^p \) for some \( z_0 \in \mathbb{C} \setminus [0, 1] \).

It is obvious that \( \lambda I - B_\delta, \lambda \in [0, 1] \), does not have bounded inverse on \( L^2 \), as it is a multiplier with symbol \( \lambda - (1 - |\xi|^2)^\delta_+ \). In the next theorem, we show that \( \lambda I - B_\delta, \lambda \in [0, 1] \), does not have bounded inverse on \( L^p \) for all \( 1 \leq p < \infty \).

**Theorem 1.2.** Let \( \delta > 0, 1 \leq p < \infty \) and \( \lambda \in [0, 1] \). Then

\[
\inf_{\|f\|_p = 1} \| (\lambda I - B_\delta) f \|_p = 0.
\]
For any $\delta > 0$, define the spectra of Bochner-Riesz mean $B_\delta$ on $L^p$ by
\[
\sigma_p(B_\delta) := \mathbb{C}\setminus\{z \in \mathbb{C}, \ zI - B_\delta \text{ has bounded inverse on } L^p\}.
\]
As Bochner-Riesz means $B_\delta$ are multiplier operators with symbols $(1 - |\xi|^2)^\delta$, we have
\[
\sigma_p(B_\delta) = \text{closure of } \{(1 - |\xi|^2)^\delta, \ \xi \in \mathbb{R}^d\} = [0, 1] \text{ for } p = 2.
\]

The equivalence in Theorem 1.1 and stability in Theorem 1.2 may not help to solve the conjecture on Bochner-Riesz means, but they imply that for any $\delta > 0$, the spectra of the Bochner-Riesz mean $B_\delta$ on $L^p$ is invariant for different $1 \leq p < \infty$ whenever it is bounded on $L^p$.

**Theorem 1.3.** Let $\delta > 0$ and $1 \leq p < \infty$. Then

(i) $\sigma_p(B_\delta) = [0, 1]$ if the Bochner-Riesz mean $B_\delta$ is bounded on $L^p$; and

(ii) $\sigma_p(B_\delta) = \mathbb{C}$ if $B_\delta$ is unbounded on $L^p$.

The above spectral invariance on different $L^p$ spaces holds for any multiplier operator $T_m$ with its bounded symbol $m$ satisfying the following hypothesis,
\[
|\xi|^k|\nabla^k m(\xi)| \in L^\infty, \ 0 \leq k \leq d/2 + 1, \tag{1.2}
\]
in the classical Mikhlin multiplier theorem, because in this case,
\[
\sigma_2(T_m) = \text{closure of } \{m(\xi), \ \xi \in \mathbb{R}^d\},
\]
and for any $z \notin \sigma_2(T_m)$, the inverse of $zI - T_m$ is a multiplier operator with symbol $(z - m(\xi))^{-1}$ satisfying (1.2) too. Inspired by the above spectral invariance for Bochner-Riesz means and Mikhlin multipliers, we propose the following problem: Under what conditions on symbol of a multiplier, does the corresponding operator have its spectrum on $L^p$ independent on $1 \leq p < \infty$.

Spectral invariance for different function spaces is closely related to algebra of singular integral operators [4, 6, 13, 23] and Wiener’s lemma for infinite matrices [10, 22, 24]. It has been established for singular integral operators with kernels being Hölder continuous and having certain off-diagonal decay [1, 6, 7, 19, 23], but it is not well studied yet for Calderon-Zygmund operators, oscillatory integrals, and many other linear operators in Fourier analysis.

In this paper, we denote by $\mathcal{S}$ and $\mathcal{D}$ the space of Schwartz functions and compactly supported $C^\infty$ functions respectively, and we use the capital letter $C$ to denote an absolute constant that could be different at each occurrence.
2 Proof of Theorem 1.1

Given nonnegative integers $\alpha_0$ and $\beta_0$, let $\mathbf{S}_{\alpha_0, \beta_0}$ contain all functions $f$ with

$$
\|f\|_{\mathbf{S}_{\alpha_0, \beta_0}} := \sum_{|\alpha| \leq \alpha_0, |\beta| \leq \beta_0} \|x^\alpha \partial^\beta f(x)\|_\infty < \infty.
$$

In this section, we prove the following strong version of Theorem 1.1.

**Theorem 2.1.** Let $\mathbf{B}$ be a Banach space of tempered distributions with $\mathbf{S}$ being dense in $\mathbf{B}$. Assume that there exist nonnegative integers $\alpha_0$ and $\beta_0$ such that any convolution operator with kernel $K \in \mathbf{S}_{\alpha_0, \beta_0}$ is bounded on $\mathbf{B}$,

$$
\|K * f\|_{\mathbf{B}} \leq C\|K\|_{\mathbf{S}_{\alpha_0, \beta_0}} \|f\|_{\mathbf{B}} \text{ for all } f \in \mathbf{B}. \quad (2.1)
$$

Then the following statements are equivalent to each other.

(i) The Bochner-Riesz mean $B_\delta$ is bounded on $\mathbf{B}$.

(ii) $(zI - B_\delta)^{-1}$ is bounded on $\mathbf{B}$ for all $z \in \mathbb{C}\setminus[0, 1]$.

(iii) $(z_0I - B_\delta)^{-1}$ is bounded on $\mathbf{B}$ for some $z_0 \in \mathbb{C}\setminus[0, 1]$.

**Proof.** (i)$\Rightarrow$(ii). Take $z \in \mathbb{C}\setminus[0, 1]$ and

$$
r_0 \in (0, \min(|z/2|^{1/\delta}, 1)/2). \quad (2.2)
$$

Let $\psi_1$ and $\psi_2 \in \mathcal{D}$ satisfy

$$
\psi_1(\xi) = 1 \text{ when } |\xi| \leq 1 - r_0, \quad \psi_1(\xi) = 0 \text{ when } |\xi| \geq 1 - r_0/2; \quad (2.3)
$$

and

$$
\psi_2(\xi) = 1 - \psi_1(\xi) \text{ if } |\xi| \leq 1 + r_0/2, \quad \psi_2(\xi) = 0 \text{ if } |\xi| > 1 + r_0. \quad (2.4)
$$

Define $m(\xi) := (z - (1 - |\xi|^2)^{\delta/2})^{-1}$, $m_1(\xi) := m(\xi)\psi_1(\xi)$ and $m_2(\xi) := m(\xi)\psi_2(\xi)$. Then $m(\xi)$ is the symbol of the multiplier operator $(zI - B_\delta)^{-1}$ and

$$
m(\xi) = m_1(\xi) + m_2(\xi) + z^{-1}(1 - \psi_1(\xi) - \psi_2(\xi)).
$$

As $m_1, \psi_1, \psi_2 \in \mathcal{D}$, multiplier operators with symbols $m_1$ and $\psi_1 + \psi_2$ are bounded on $\mathbf{B}$ by (2.1). Therefore the proof reduces to establishing the boundedness of the multiplier operator with symbol $m_2$,

$$
\|(m_2 \hat{f})^\vee\|_{\mathbf{B}} \leq C\|f\|_{\mathbf{B}}, \quad f \in \mathbf{B}, \quad (2.5)
$$

where \( f^\vee \) is the inverse Fourier transform of \( f \).

Take an integer \( N_0 > \alpha_0/\delta \). Write

\[
m_2(\xi) = z^{-1}\left( \sum_{n=0}^{N_0} + \sum_{n=N_0+1}^{\infty} \right)(z^{-1})^n((1 - |\xi|^2)^\delta)\psi_2(\xi) =: m_{21}(\xi) + m_{22}(\xi),
\]

and denote multiplier operators with symbols \( m_{21} \) and \( m_{22} \) by \( T_{21} \) and \( T_{22} \) respectively. Observe that

\[
T_{21} = z^{-1}\psi_2 + \sum_{n=1}^{N_0} z^{-n-1}(B\delta)^n\psi_2,
\]

where \( \psi_2 \) is the multiplier operator with symbol \( \psi_2 \). Then \( T_{21} \) is bounded on \( B \) by (2.1) and the boundedness assumption (i),

\[
\|T_{21}f\|_B \leq C\|f\|_B \quad \text{for all } f \in B. \tag{2.6}
\]

Recall that \( \psi_2 \in D \) is supported on \( \{ \xi, 1 - r_0 \leq |\xi| \leq 1 + r_0 \} \). Then the inverse Fourier transform \( K_n \) of \( (1 - |\xi|^2)^\delta \psi_2(\xi) \) satisfies

\[
\|K_n\|_{S_{\alpha_0,\beta_0}} \leq Cn^{\alpha_0}(2r_0)^n, \quad n \geq N_0 + 1.
\]

Therefore the convolution kernel

\[
K(x) := z^{-1}\sum_{n=N_0+1}^{\infty} z^{-n}K_n(x)
\]

of \( T_{22} \) belongs to \( S_{\alpha_0,\beta_0} \). This together with (2.1) proves

\[
\|T_{22}f\|_B \leq C\|f\|_B \quad \text{for all } f \in B. \tag{2.7}
\]

Combining (2.6) and (2.7) proves (2.5) and hence completes the proof of the implication (i) \( \Rightarrow \) (ii).

(ii) \( \Rightarrow \) (iii). The implication is obvious.

(iii) \( \Rightarrow \) (i). Let \( z_0 \in C\setminus[0,1] \) so that \((z_0I - B\delta)^{-1}\) is bounded on \( B \), \( r_0 \) be as in (2.2) with \( z \) replaced by \( z_0 \), and let \( \psi_1, \psi_2 \) be given in (2.3) and (2.4) respectively. Following the argument used in the proof of the implication (i) \( \Rightarrow \) (ii), we see that it suffices to prove the operator \( T_3 \) associated with
the multiplier \( m_3(\xi) := (1 - |\xi|^2)_{+}^{\delta} \psi_2(\xi) \) is bounded on \( B \). Take an integer \( N_0 > \alpha_0/\delta \) and write

\[
m_3(\xi) = -z_0 \left( \sum_{n=1}^{N_0} + \sum_{n=N_0+1}^{\infty} \right) (1-\frac{z_0}{(1-|\xi|^2)_{+}^{\delta}})^n \psi_2(\xi) =: m_{31}(\xi) + m_{32}(\xi),
\]

where the series is convergent since

\[
|1 - z_0(1 - |\xi|^2)_{+}^{\delta} - 1| \leq \sum_{n=1}^{\infty} \left( |z_0|^{-1} (1 - |\xi|^2)_{+}^{\delta} \right)^n \leq \sum_{n=1}^{\infty} \left| z_0 \right|^{-1} (2r_0)^{\delta} n < 1.
\]

Denote by \( T_{31} \) and \( T_{32} \) the operators associated with multiplier \( m_{31} \) and \( m_{32} \) respectively. As \( T_{31} \) is a linear combination of \( \Psi_2 \) and \( (z_0 I - B_{\delta})^{-n}\Psi_2, 1 \leq n \leq N_0 \), it is bounded on \( B \),

\[
\|T_{31}f\|_B \leq C\|f\|_B \quad \text{for all } f \in B, \tag{2.8}
\]

by (2.1) and the boundedness assumption (iii).

Define the inverse Fourier transform of \( (1-\frac{z_0}{(1-|\xi|^2)_{+}^{\delta}})^n \psi_2(\xi), n > N_0 \), by \( \tilde{K}_n \). One may verify that

\[
\|\tilde{K}_n\|_{S_{\alpha_0,0}} \leq Cn^{\alpha_0}(2r_0)^{nd}\left| z_0 \right|^{-n}, n \geq N_0 + 1.
\]

Therefore

\[
\|T_{32}f\|_B \leq C \sum_{n=N_0+1}^{\infty} \|\tilde{K}_n\|_{S_{\alpha_0,0}} \|f\|_B \leq C \left( \sum_{n=N_0+1}^{\infty} n^{\alpha_0}(2r_0)^{nd}\left| z_0 \right|^{-n} \right) \|f\|_B \tag{2.9}
\]

for all \( f \in B \). Combining (2.8) and (2.9) completes the proof. \( \square \)

### 3 Proof of Theorem 1.2

Let \( f \) and \( K \) be Schwartz functions with \( f(0) = 1 \) and \( \hat{K}(0) = 0 \), and set \( f_N(x) = N^{-d}f(x/N), N \geq 1 \). Then for any positive integer \( \alpha \geq d + 1 \) there
exists a constant $C_\alpha$ such that

\[
|K * f_N(x)| \leq \left( \int_{|x-y|>\sqrt{N}} + \int_{|x-y|\leq\sqrt{N}} \right) |K(x-y)| \ |f_N(y) - f_N(x)|\,dy
\]

\[
\leq \int_{|x-y|>\sqrt{N}} |K(x-y)| \ |f_N(y)|\,dy + C_\alpha N^{-d-1/2}(1 + |x/N|)^{-\alpha}
\]

\[
\leq C_\alpha N^{-1/2} \left( \int_{\mathbb{R}^d} (1 + |x-y|)^{-\alpha} |f_N(y)|\,dy + N^{-d}(1 + |x/N|)^{-\alpha} \right).
\]

(3.1)

This implies that

\[
\lim_{N \to \infty} \frac{\|K * f_N\|_p}{\|f_N\|_p} = 0, \ 1 \leq p < \infty.
\]

Therefore the following is a strong version of Theorem 1.2.

**Theorem 3.1.** Let $\mathcal{B}$ be a Banach space of tempered distributions with $\mathcal{S}$ being dense in $\mathcal{B}$. Assume that (2.1) holds for some $\alpha_0, \beta_0 \geq 0$ and that for any $\xi_0 \in \mathbb{R}^d$ there exists $\varphi_0 \in \mathcal{D}$ such that $\hat{\varphi}_0(0) = 1$ and

\[
\lim_{N \to \infty} \frac{\|(m\hat{f}_{N,\xi_0})^\vee\|_{\mathcal{B}}}{\|f_{N,\xi_0}\|_{\mathcal{B}}} = 0
\]

for all Schwartz functions $m$ with $m(\xi_0) = 0$, where $\hat{f}_{N,\xi_0}(\xi) = \varphi_0(N(\xi - \xi_0)).$

Then

\[
\inf_{f \neq 0} \frac{\|(\lambda I - B_\delta)f\|_{\mathcal{B}}}{\|f\|_{\mathcal{B}}} = 0 \quad \text{for all } \lambda \in [0, 1].
\]

**Proof.** The infimum in (3.3) is obvious for $\lambda = 0$. So we assume that $\lambda \in (0, 1]$ from now on. Select $\xi_0 \in \mathbb{R}^d$ so that $1 - |\xi_0|^2 \geq 1$. Then for sufficiently large $N \geq 1$,

\[
(\lambda I - B_\delta)f_{N,\xi_0} = (m_{\xi_0} \hat{f}_{N,\xi_0})^\vee,
\]

(3.4)

where $m_{\xi_0}(\xi) = (\lambda - (1 - |\xi|^2)^\vee) \psi(\xi - \xi_0)$ and $\psi \in \mathcal{D}$ is so chosen that $\psi(\xi) = 1$ for $|\xi| \leq (1 - |\xi_0|)/2$ and $\psi(\xi) = 0$ for $|\xi| \geq 1 - |\xi_0|$. Observe that $m_{\xi_0} \in \mathcal{D}$ satisfies $m_{\xi_0}(\xi_0) = 0$. This together with (3.2) and (3.4) proves that

\[
\lim_{N \to \infty} \frac{\|(\lambda I - B_\delta)f_{N,\xi_0}\|_{\mathcal{B}}}{\|f_{N,\xi_0}\|_{\mathcal{B}}} = 0.
\]

Hence (3.3) is proved for $\lambda \in (0, 1]$. \hfill \Box

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Remark 3.2. For $\xi \in \mathbb{R}^d$, define modulation operator $M_\xi$ by

$$M_\xi f(x) = e^{ix\xi}f(x).$$

We say that a Banach space $B$ is modulation-invariant if for any $\xi \in \mathbb{R}^d$ there exists a positive constant $C_\xi$ such that

$$\|M_\xi f\|_B \leq C_\xi \|f\|_B, \quad f \in B.$$ 

Such a Banach space with modulation bound $C_\xi$ being dominated by a polynomial of $\xi$ was introduced in [26] to study oscillatory integrals and Bochner-Riesz means. Modulation-invariant Banach spaces include weighted $L^p$ spaces, Triebel-Lizorkin spaces $F^\alpha_{p,q}$, Besov spaces $B^\alpha_{p,q}$, Herz spaces $K^\alpha_{p,q}$, and modulation spaces $M^p_{p,q}$, where $\alpha \in \mathbb{R}$ and $1 \leq p, q < \infty$ [9, 11, 16, 25]. For functions $f_{N,\xi_0}, N \geq 1$, in Theorem 3.1,

$$f_{N,\xi_0}(x) = e^{ix\xi_0}(\varphi_0(N \cdot))^{\vee}(x)$$

and

$$(m\widehat{f_{N,\xi_0}})^{\vee}(x) = e^{ix\xi_0}(m_{\xi_0}\varphi_0(N \cdot))^{\vee}(x),$$

where $m_{\xi_0}(\xi) = m(\xi + \xi_0)$ satisfies $m_{\xi_0}(0) = 0$. Then for a modulation-invariant space $B$, the limit (3.2) holds for any $\xi_0 \in \mathbb{R}^d$ if and only if it is true for $\xi_0 = 0$. Therefore we obtain the following result from Theorems 2.1 and 3.1.

Corollary 3.3. Let $B$ be a modulation-invariant Banach space of tempered distributions with $\mathcal{S}$ being dense in $B$. Assume that (2.1) holds for some $\alpha_0, \beta_0 \geq 0$ and that there exists $\varphi_0 \in \mathcal{D}$ such that

$$\hat{\varphi}_0(0) = 1 \text{ and } \lim_{N \to \infty} \|(m\varphi_0(N \cdot))^{\vee}\|_B/\|\varphi_0(N \cdot)^{\vee}\|_B = 0$$

for all Schwartz functions $m(\xi)$ with $m(0) = 0$. If the Bochner-Riesz mean $B_\delta$ is bounded on $B$, then its spectrum on $B$ contains the unit interval $[0, 1]$.

4 Remarks

In this section, we extend conclusions in Theorem 1.3 to weighted $L^p$ spaces, Triebel-Lizorkin spaces, Besov spaces, and Herz spaces.
4.1 Spectra on weighted $L^p$ spaces

Let $1 \leq p < \infty$ and $Q$ contain all cubes $Q \subset \mathbb{R}^d$. A positive function $w$ is said to be a Muckenhoupt $A_p$-weight if

$$
\left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq C, \ Q \in Q
$$

for $1 < p < \infty$, and

$$
\frac{1}{|Q|} \int_Q w(x) \, dx \leq C \inf_{x \in Q} w(x), \ Q \in Q
$$

for $p = 1$ [8, 17]. For $\delta > (d - 1)/2$, convolution kernel of the Bochner-Riesz mean $B_\delta$ is dominated by a multiple of $(1 + |x|)^{-\delta - (d+1)/2}$ and hence it is bounded on weighted $L^p$ space $L^p_w$ for all $1 \leq p < \infty$ and Muckenhoupt $A_p$-weights $w$. For $\delta = (d - 1)/2$, complex interpolation method was introduced in [18] to establish $L^p_w$-boundedness of $B_\delta$ for all $1 < p < \infty$ and Muckenhoupt $A_p$-weights $w$. The reader may refer to [8, 15] and references therein for $L^p_w$-boundedness of Bochner-Riesz means with various weights $w$. In this subsection, we consider spectra of Bochner-Riesz means on $L^p_w$.

**Theorem 4.1.** Let $\delta > 0, 1 \leq p < \infty,$ and $w$ be a Muckenhoupt $A_p$-weight. If the Bochner-Riesz mean $B_\delta$ is bounded on $L^p_w$, then its spectrum on $L^p_w$ is the unit interval $[0, 1]$.

**Proof.** Denote the norm on $L^p_w$ by $\| \cdot \|_{p,w}$. By Theorems 2.1 and 3.1, and modulation-invariance of $L^p_w$, it suffices to prove

$$
\|K \ast f\|_{p,w} \leq C\|K\|_{S^{d+1,0}} \|f\|_{p,w} \text{ for all } f \in L^p_w, \tag{4.1}
$$

and

$$
\lim_{N \to \infty} \| (m \varphi_0(N \cdot))^\vee \|_{p,w} = 0 \tag{4.2}
$$

for all Schwartz functions $\varphi_0$ and $m$ with $\varphi_0(0) = 1$ and $m(0) = 0$.

Observe that $|K(x)| \leq C\|K\|_{S^{d+1,0}} (1 + |x|)^{-d-1}$. Then (4.1) follows from the standard argument for weighted norm inequalities [8].

Recall that any $A_p$-weight is a doubling measure [8]. This doubling property together with (3.1) leads to

$$
\|(m \varphi_0(N \cdot))^\vee\|_{p,w}^p \leq C N^{-p/2} \|(\varphi_0(N \cdot))^\vee\|_{p,w}^p + C N^{-(d+1)/2} p w([-N, N]^d). \tag{4.3}
$$
On the other hand, there exists $\epsilon_0 > 0$ such that $|\varphi_0'(x)| \geq |\varphi_0'(0)|/2 \neq 0$ for all $|x| \leq \epsilon_0$. This implies that

$$\|((\varphi_0 N\cdot))')\|_{p,w} \geq CN^{-d/p} w([-\epsilon_0 N, \epsilon_0 N]^d).$$

(4.4)

Combining (4.3), (4.4) and the doubling property for the weight $w$, we establish the limit (4.2) and complete the proof.

### 4.2 Spectra on Triebel-Lizorkin spaces and Besov spaces

Let $\phi_0$ and $\psi \in \mathcal{S}$ be so chosen that $\hat{\phi_0}$ is supported in $\{\xi, |\xi| \leq 2\}$, $\hat{\psi}$ supported in $\{\xi, 1/2 \leq |\xi| \leq 2\}$, and

$$\hat{\phi_0}(\xi) + \sum_{l=1}^{\infty} \hat{\psi}(2^{-l}\xi) = 1, \quad \xi \in \mathbb{R}^d.$$

For $\alpha \in \mathbb{R}$ and $1 \leq p, q < \infty$, let Triebel-Lizorkin space $F_{p,q}^\alpha$ contain all tempered distributions $f$ with

$$\|f\|_{F_{p,q}^\alpha} := \|\phi_0 * f\|_p + \left\| \left( \sum_{l=1}^{\infty} 2^{l\alpha q} |\psi_l * f|^q \right)^{1/q} \right\|_p < \infty,$$

where $\psi_l = 2^{ld} \psi(2^l \cdot), l \geq 1$. Similarly, let Besov space $B_{p,q}^\alpha$ be the space of tempered distributions $f$ with

$$\|f\|_{B_{p,q}^\alpha} := \|\phi_0 * f\|_p + \left( \sum_{l=1}^{\infty} 2^{l\alpha q} \|\psi_l * f\|_p^q \right)^{1/q} < \infty.$$

Next is our results about spectra of Bochner-Riesz means on Triebel-Lizorkin spaces and on Besov spaces.

**Theorem 4.2.** Let $\delta > 0, \alpha \in \mathbb{R}$ and $1 \leq p, q < \infty$. If the Bochner-Riesz mean $B_{\delta}$ is bounded on $F_{p,q}^\alpha$ (resp. $B_{p,q}^\alpha$), then its spectrum on $F_{p,q}^\alpha$ (resp. on $B_{p,q}^\alpha$) is the unit interval $[0, 1]$.

**Proof.** For $z \not\in [0, 1]$ and $\delta > 0$, both $B_{\delta}$ and $(zI - B_{\delta})^{-1} - z^{-1}I$ are multiplier operators with compactly supported symbols. Therefore $B_{\delta}$ (resp. $(zI - B_{\delta})^{-1}$) is bounded on the Triebel-Lizorkin space $F_{p,q}^\alpha$ if and only if it is bounded on the Besov space $B_{p,q}^\alpha$ if and only if it is bounded on $L^p$. The above equivalence together with Theorem 1.1 yields our desired conclusions. \qed
4.3 Spectra on Herz spaces

For $\alpha \in \mathbb{R}$ and $1 \leq p, q < \infty$, let Herz space $K^{\alpha,q}_p$ contain all locally $p$-integrable functions $f$ with

$$
\|f\|_{K^{\alpha,q}_p} := \|f \chi_{|\cdot| \leq 1}\|_p + \left(\sum_{l=1}^{\infty} 2^{lq} \|f \chi_{2^{l-1} < |\cdot| \leq 2^l}\|_p\right)^{1/q} < \infty,
$$

where $\chi_E$ is the characteristic function on a set $E$. The boundedness of Bochner-Riesz means on Herz spaces is well studied, see for instance [16, 30]. Following the argument used in the proof of Theorem 4.1, we have

**Theorem 4.3.** Let $\delta > 0$, $1 \leq p, q < \infty$ and $\alpha > -d/p$. If the Bochner-Riesz mean $B_\delta$ is bounded on $K^{\alpha,q}_p$, then its spectrum on $K^{\alpha,q}_p$ is $[0, 1]$.

Acknowledgements

The project is partially supported by National Science Foundation (DMS-1412413) and NSF of China (Grant Nos. 11426203).

References


