Standing Waves in Discrete Inhomogeneous Media

Brian E. Moore
bmoore@math.ucf.edu

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Bistable Waves in Discrete Inhomogeneous Media

Collaborators
- Tony Humphries, McGill University
- Erik Van Vleck, University of Kansas

Students
- Lory Ajamian, McGill University
- Jessica Long, University of Iowa
- Joe Segal, University of Central Florida
- Nicole Lopez, University of Central Florida

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Our Nervous System

The nervous cells live inside the “Hot Dog Buns” which are called myelin sheath.

The inrush of sodium ($Na^+$) at the sodium channels causes the electric impulse to jump to the next cell.

Multiple Sclerosis causes the destruction of myelin, which helps carry electrical signals.
Electrical Impulse with Diseased Cells

Questions

- Where does the electrical impulse stop?
- How much cell destruction is required to make it stop?
Bistable Equation with Inhomogeneous Diffusion

\[ \dot{u}_j = \alpha_j (u_{j+1} - u_j) - \alpha_{j-1} (u_j - u_{j-1}) - f(u_j) \]

with

\[ \alpha_j = \begin{cases} 
\alpha_j & -m \leq j \leq n \\
\alpha & j < -m \text{ or } j > n 
\end{cases} \]

\( m, n \in \{0\} \cup \mathbb{N} \)

The nonlinearity is the derivative of a double-well potential,

\[ f(u) = u(u - a)(u - 1) \text{ with } a \in (0, 1). \]
McKean’s Caricature of the Cubic

Dashed red line:

\[ f(u) = u - h(u - a) \]

\[ h(x) = \begin{cases} 
1 & x > 0 \\
[0,1] & x = 0 \\
0 & x < 0 
\end{cases} \]
Numerical Simulations for the Evolution Equation

For the case of a single defect

\[ \alpha_j = \begin{cases} 
0.6 & j = 30 \\
1 & j \neq 30 
\end{cases} \]

A slightly slower wave is stopped by the defect.
Problem History

- Cahn, Mallet-Paret, and Van Vleck (1999) Derive traveling wave solutions on 2-D lattice with $\alpha_j = \alpha$ and discuss relationship between $\alpha$ and the wave speed.

- Lewis and Keener (2000) steady states of the PDE model Changing parameters $m$, $n$, and $\alpha_j$ leads to steady state solutions through a limit point bifurcation.

- Elmer and Van Vleck (2001) periodic diffusion Derive solutions using piecewise linear $f(u)$, and discuss changes in the wave speed as the solution evolves.
**Steady State Solutions**

**Definition**: The range of \( a \) values that yield standing waves is called the *interval of propagation failure*.

Standing waves are solutions of \( \dot{u}_j = 0 \) or equivalently

\[
\alpha_j (u_{j+1} - u_j) - \alpha_{j-1} (u_j - u_{j-1}) = f(u_j)
\]

\[
\lim_{j \to \infty} u_j = 1 \quad \lim_{j \to -\infty} u_j = 0
\]

where \( f(u_j) = u_j - h(u_j - a) \).

Note: Solutions are not translationally invariant.
Families of Standing Waves

Define $\xi^* \in \mathbb{R}$ a parameter that determines the position of the wave relative to the defect region.
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Define $j^* = \lfloor \xi^* \rfloor$ so that

$$u_{j^*} = a \text{ or } u_{j^*} < a < u_{j^*+1}$$
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We seek solutions that satisfy

$$u_j < a \quad \text{for} \quad j < \xi^* \quad \text{and} \quad u_j > a \quad \text{for} \quad j > \xi^*.$$
Families of Standing Waves

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Define $j* = \lfloor \xi* \rfloor$ so that

$$u_{j*} = a \quad \text{or} \quad u_{j*} < a < u_{j*} + 1$$

We seek solutions that satisfy

$$u_j < a \quad \text{for} \quad j < \xi^* \quad \text{and} \quad u_j > a \quad \text{for} \quad j > \xi^*.$$

$$\Rightarrow \quad f(u_j) = u_j - h(u_j - a) = u_j - h(j - \xi^*)$$
Derivation of Solutions

We use Jacobi operator theory (Teschl 2000) to solve the difference equation

\[ \alpha_j (u_{j+1} - u_j) - \alpha_{j-1} (u_j - u_{j-1}) - u_j = -h_j \]

for

\[ h_j = \begin{cases} 1 & j > j^* \\ h_{j^*} & j = j^* \\ 0 & j < j^* \end{cases} \]

where

\[ h_{j^*} = \begin{cases} [0,1] & \xi^* = j^* \\ 0 & j^* < \xi^* < j^* + 1 \end{cases} \]
General Solution = Homogeneous Solution + Particular Solution

\[ u_j = u_{j^*} \rho_j + u_{j^*+1} \sigma_j + \begin{cases} \sum_{k=j^*+1}^{j} h_k \sigma_{j-k} & \text{if } j > j^* \\ 0 & \text{if } j = j^* \\ \frac{h_{j^*}}{\alpha_{j^*}} \sigma_{j-j^*} & \text{if } j < j^* \end{cases} \]

\( \rho_j \) and \( \sigma_j \) are fundamental solutions and may be constructed recursively using

\[ \alpha_j(u_{j+1} - u_j) - \alpha_{j-1}(u_j - u_{j-1}) - u_j = 0 \]

with

\[ \rho_{j^*} = 1, \quad \rho_{j^*+1} = 0, \quad \sigma_{j^*} = 0, \quad \sigma_{j^*+1} = 1. \]

The particular solution can be found in Teschl (2000).
Standing Waves for

\[ \alpha_j (u_{j+1} - u_j) - \alpha_{j-1} (u_j - u_{j-1}) - u_j = -h_j \]

**Theorem** (Humphries, Moore, Van Vleck 2010)

Suppose

\[ \alpha_j = \begin{cases} 
\alpha_j & -m \leq j \leq n \\
\alpha & j < -m \text{ or } j > n 
\end{cases} \]

and that \( j^* \in [-m, n] \). Then the existence of solutions is guaranteed by the necessary and sufficient conditions:

1. \( a \in (u_{j^*}, u_{j^*+1}) \) with \( h_{j^*} = 0 \), if \( j^* < \xi^* < j^* + 1 \)
2. \( a \in u_{j^*} \) with \( h_{j^*} = [0, 1] \), if \( j^* = \xi^* \).

Explicit formulae for these solutions are a bit too complicated for this slide, but we have them, I promise.
Standing Waves for $\alpha_j = \alpha, \forall j$

$\xi^* = 0$ implies $a = u_0$, giving an unstable wave

$$u_j = \begin{cases} 
1 + (a - 1)\lambda^{-j} & j \geq 0 \\
a\lambda^j & j \leq 0.
\end{cases}$$

$a \in [v_0, w_0]$. 

$0 < \xi^* < 1$ implies $u_0 < a < u_1$, giving a stable wave

$$u_j = \begin{cases} 
1 - v_0\lambda^{1-j} & j \geq 0 \\
v_0\lambda^j & j \leq 0.
\end{cases}$$

$a \in (v_0, w_0)$

where $\lambda = (1 + \sqrt{1 + 4\alpha})/2\alpha$,

$$v_0 = \frac{1}{\lambda + 1} \quad \text{and} \quad w_0 = \frac{\lambda}{\lambda + 1}$$
Standing Waves for $\alpha_j = \alpha$, $\forall j$
Standing Waves for $\alpha = 1$ and $\xi^* \in (0, 1)$
circles: $\alpha_0 = 1$, crosses: $\alpha_0 = 1/4$
Interval of Propagation Failure

\[-3 < \xi^* < -2\]

\[-2 < \xi^* < -1\]

\[-1 < \xi^* < 0\]

\[0 < \xi^* < 1\]
Interval of Propagation Failure

**Theorem** (Humphries, Moore, Van Vleck 2010)

If $a$ yields a traveling wave for $\alpha_0 = \alpha$, then

- Either $a \in (0, 1/(\lambda + 2))$ or $a \in ((\lambda + 1)/(\lambda + 2), 1)$ with

  $$\lambda = (1 + \sqrt{1 + 4\alpha})/2\alpha$$

- There are no corresponding standing waves for $\alpha_0 < \alpha$ and $\xi^* \notin (0, 1)$, nor for $\alpha_0 > \alpha$ and $\xi^* \in (0, 1)$.

- There exist standing waves for $\alpha_0 < \alpha$ and $\xi^* \in (0, 1)$, and for $\alpha_0 > \alpha$ and $\xi^* \notin (0, 1)$, provided

  $$a \in \left[ \frac{\alpha_0/\alpha}{\lambda + 2(\alpha_0/\alpha)}, \frac{\lambda + \alpha_0/\alpha}{\lambda + 2(\alpha_0/\alpha)} \right].$$
Interval of Propagation Failure

\[ \alpha = 1 \]
Interval of Propagation Failure

\[ \alpha = 1 \]

\[ \alpha_0 = 5 \]

\[ a \]

\[ \xi^* \]
Interval of Propagation Failure

**Blue:** $\alpha_0 = \alpha = 1/2$  \hspace{1cm} **Black:** $\alpha_0 = 1/2, \alpha = 2$
Interval of Propagation Failure

\[ \alpha = 1, \quad \alpha_{\text{defect}} = 0.2 \]
Spatially Discrete FitzHugh-Nagumo Equations

\[ \dot{u}_j = \alpha_j(u_{j+1} - u_j) - \alpha_{j-1}(u_j - u_{j-1}) - v_j - f(u_j, a), \]
\[ \dot{v}_j = bu_j - rv_j \]

The recovery term \( v_j \) gives pulse solutions.

Standing waves \( \Rightarrow \) \( v_j = \frac{b}{r}u_j \)

\[ \Rightarrow \quad \alpha_j(u_{j+1} - u_j) - \alpha_{j-1}(u_j - u_{j-1}) - \frac{b}{r}u_j = f(u_j, a) \]

with boundary conditions

\[ \lim_{j \to \pm\infty} u_j = 0 \]
Spatially Discrete FitzHugh-Nagumo Equations

We seek solutions that satisfy

\[ u_j < a \quad \text{for} \quad j < \xi^* \quad \text{and} \quad j > \xi^{**} \]

and

\[ u_j > a \quad \text{for} \quad \xi^* < j < \xi^{**}. \]

Rewriting the piecewise linear \( f \), yields the equation

\[ \alpha_j(u_{j+1} - u_j) - \alpha_{j-1}(u_j - u_{j-1}) - \left(1 + \frac{b}{r}\right)u_j = g_j \]

where \( j^* = \lfloor \xi^* \rfloor \) and \( j^{**} = \lceil \xi^{**} \rceil \) implies

\[ g_j = h(j - j^*) - h(j - j^{**}) \]
FHN Waves and Intervals of Propagation Failure
Conclusion

Summary of Results

- General formula for solutions and intervals of propagation failure are provided.
- Propagation failure can only occur in a specific interval near the defects.
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- General formula for solutions and intervals of propagation failure are provided.
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Future Work

- Non-monotonic waves (multiple crossings of $a$)
- 2-dimensional lattice
- Other nonlinearities