Spectra of Bochner-Riesz means on $L^p$

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Abstract
The Bochner-Riesz mean $B_{\delta, \delta}$, $\delta > 0$, is shown to have the unit interval $[0, 1]$ as its spectrum on $L^p$ when it is bounded on $L^p$.

1 Introduction and Main Results
Define Fourier transform $\hat{f}$ of an integrable function $f$ by

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-ix\xi}dx$$

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and extend its definition to all tempered distributions as usual. Consider Bochner-Riesz means $B_{\delta}, \delta > 0$, on $\mathbb{R}^d$,

$$\widehat{B_{\delta}f}(\xi) := (1 - |\xi|^2)^{\delta}_+ \hat{f}(\xi),$$

where $t_+ = \max(t, 0), t \in \mathbb{R}$ [Bo, Gr, St1, SW]. A conjecture is that the Bochner-Riesz mean $B_{\delta}$ is bounded on $L^p$, the space of all $p$-integrable functions on $\mathbb{R}^d$, if and only if

$$\delta > \left( d \left| \frac{1}{p} - \frac{1}{2} \right| - \frac{1}{2} \right)_+, \ 1 \leq p < \infty. \quad (1.1)$$

The requirement (1.1) on the index $\delta$ is necessary for $L^p$ boundedness of the Bochner-Riesz mean $B_{\delta}$ [H]. The sufficiency is completely solved only for dimension two [CP] and it is still open for high dimensions, see [Bou, L, St2, TV1, TV2, TVV, Wo] and references therein.

Denote the identity operator by $I$. Consider spectra

$$\sigma_p(B_{\delta}) := \mathbb{C} \setminus \{ z \in \mathbb{C}, \ zI - B_{\delta} \text{ has bounded inverse on } L^p \}, \ 1 \leq p < \infty,$$

of Bochner-Riesz means $B_{\delta}, \delta > 0$, on $L^p$. For $p = 2$,

$$\sigma_2(B_{\delta}) = \text{closure of } \{(1 - |\xi|^2)^{\delta}_+, \ \xi \in \mathbb{R}^d\} = [0, 1],$$

as Bochner-Riesz means $B_{\delta}$ are multiplier operators with symbols $(1 - |\xi|^2)^{\delta}_+$. For $p \neq 2$, we have the following result.

**Theorem 1.1.** Let $\delta > 0$ and $1 \leq p < \infty$. If the Bochner-Riesz mean $B_{\delta}$ is bounded on $L^p$, then its spectrum $\sigma_p(B_{\delta})$ on $L^p$ is the unit interval $[0, 1]$.

From Theorem 1.1 we see that spectra of Bochner-Riesz means on different $L^p$ spaces are the same.

The above spectral invariance on different $L^p$ spaces holds for any multiplier operator $T_m$ with its bounded symbol $m$ satisfying the following hypothesis,

$$|\xi|^k |\nabla^k m(\xi)| \in L^\infty, \ 0 \leq k \leq d/2 + 1, \quad (1.2)$$

in the classical Mikhlin multiplier theorem, because in this case,

$$\sigma_2(T_m) = \text{closure of } \{ m(\xi), \ \xi \in \mathbb{R}^d \},$$

and for any $\lambda \not\in \sigma_2(T_m)$, the inverse of $\lambda I - T_m$ is a multiplier operator with symbol $(\lambda - m(\xi))^{-1}$ satisfying (1.2) too. Based on the above observations,
we propose the following problem for multiplier operators: Under what conditions on symbol \( m \) does the corresponding multiplier operator \( T_m \) have its spectrum on \( L^p \) independent on \( 1 \leq p < \infty \).

Spectral invariance for different function spaces is closely related to algebra of singular integral operators \([\text{CZ, FS, K, Su2}]\) and Wiener’s lemma for infinite matrices \([\text{GL, Su1, Su3}]\). It has been established for singular integral operators with kernels being Hölder continuous and having certain off-diagonal decay \([\text{Ba, FS, FSS, ShS, Su2}]\), but it is not well studied yet for Calderon-Zygmund operators, oscillatory integrals, and many other linear operators in Fourier analysis.

In this paper, we denote by \( \mathcal{S} \) and \( \mathcal{D} \) the space of Schwartz functions and compactly supported \( C^\infty \) functions respectively, and we use the capital letter \( C \) to denote an absolute constant that may be different at each occurrence.

## 2 Proofs

The proof of Theorem 1.1 reduces to showing

\[(2.1) \quad \sigma_p(B_\delta) \subset [0, 1]\]

and

\[(2.2) \quad [0, 1] \subset \sigma_p(B_\delta).\]

Denote by \( \| \cdot \|_p \) the norm on \( L^p, 1 \leq p \leq \infty \). Given nonnegative integers \( \alpha_0 \) and \( \beta_0 \), let \( \mathcal{S}_{\alpha_0, \beta_0} \) contain all functions \( f \) with

\[\|f\|_{\mathcal{S}_{\alpha_0, \beta_0}} := \sum_{|\alpha| \leq \alpha_0, |\beta| \leq \beta_0} \|x^\alpha \partial^\beta f(x)\|_\infty < \infty.\]

The inclusion (2.1) follows from the following theorem.

**Theorem 2.1.** Let \( \mathcal{B} \) be a Banach space of tempered distributions with \( \mathcal{S} \) being dense in \( \mathcal{B} \). Assume that there exist nonnegative integers \( \alpha_0 \) and \( \beta_0 \) such that any convolution operator with kernel \( K \in \mathcal{S}_{\alpha_0, \beta_0} \) is bounded on \( \mathcal{B} \),

\[(2.3) \quad \|K \ast f\|_\mathcal{B} \leq C\|K\|_{\mathcal{S}_{\alpha_0, \beta_0}} \|f\|_\mathcal{B} \quad \text{for all} \quad f \in \mathcal{B}.\]

If the Bochner-Riesz mean \( B_\delta \) is bounded on \( \mathcal{B} \), then \( (\lambda I - B_\delta)^{-1} \) is bounded on \( \mathcal{B} \) for any \( \lambda \in \mathbb{C}\setminus[0, 1] \).
Proof. Take $\lambda \in \mathbb{C}[0, 1]$ and $r_0 \in (0, \min(\{1 \[1/\delta, 1\})/2)$. Let $\psi_1$ and $\psi_2 \in \mathcal{D}$ satisfy

$$
\psi_1(\xi) = 1 \text{ when } |\xi| \leq 1 - r_0, \psi_1(\xi) = 0 \text{ when } |\xi| \geq 1 - r_0/2;
$$

and

$$
\psi_2(\xi) = 1 - \psi_1(x) \text{ if } |\xi| \leq 1 + r_0/2, \psi_2(\xi) = 0 \text{ if } |\xi| > 1 + r_0.
$$

Define $m(\xi) := (\lambda - (1 - |\xi|^2)^{\delta})^{-1}$, $m_1(\xi) := m(\xi)\psi_1(\xi)$ and $m_2(\xi) := m(\xi)\psi_2(\xi)$. Then $m(\xi)$ is the symbol of the multiplier operator $(\lambda I - B_\delta)^{-1}$ and

$$
m(\xi) = m_1(\xi) + m_2(\xi) + \lambda^{-1}(1 - \psi_1(\xi) - \psi_2(\xi)).
$$

As $m_1, \psi_1, \psi_2 \in \mathcal{D}$, multiplier operators with symbols $m_1$ and $\psi_1 + \psi_2$ are bounded on $B$ by (2.3). Therefore the proof reduces to establishing the boundedness of the multiplier operator with symbol $m_2$.

(2.4) \quad \|m(\hat{f})\|_B \leq C\|f\|_B, \ f \in B.

where $f^\vee$ is the inverse Fourier transform of $f$.

Take an integer $N_0 > \alpha_0/\delta$. Write

$$
m_2(\xi) = \lambda^{-1}\left(\sum_{n=0}^{N_0} + \sum_{n=N_0+1}^{\infty}\right)(\lambda^{-1})^n((1 - |\xi|^2)^{\delta})^n\psi_2(\xi) =: m_{21}(\xi) + m_{22}(\xi),
$$

and denote multiplier operators with symbols $m_{21}$ and $m_{22}$ by $T_{21}$ and $T_{22}$ respectively. Observe that

$$
T_{21} = \lambda^{-1}\Psi_2 + \sum_{n=1}^{N_0} \lambda^{-n-1}(B_\delta)^n\Psi_2,
$$

where $\Psi_2$ is the multiplier operator with symbol $\psi_2$. Then $T_{21}$ is bounded on $B$ by (2.3) and boundedness assumption for the Bochner-Riesz mean $B_\delta$,

(2.5) \quad \|T_{21}f\|_B \leq C\|f\|_B \quad \text{for all } f \in B.

Recall that $\psi_2 \in \mathcal{D}$ is supported on $\{\xi, 1 - r_0 \leq |\xi| \leq 1 + r_0\}$. Then the inverse Fourier transform $K_n$ of $(1 - |\xi|^2)^{\delta} \psi_2(\xi)$ satisfies

$$
\|K_n\|_{s_{\alpha_0/\delta}} \leq Cn^{\alpha_0}(2r_0)^{n\delta}, \ n \geq N_0 + 1.
$$

Therefore the convolution kernel

$$
K(x) := \lambda^{-1}\sum_{n=N_0+1}^{\infty}\lambda^{-n}K_n(x)
$$
of $T_{22}$ belongs to $S_{\alpha_0, \beta_0}$. This together with (2.3) proves

(2.6) $\|T_{22}f\|_B \leq C\|f\|_B$ for all $f \in B$.

Combining (2.5) and (2.6) proves (2.4). This completes the proof.

Let $f$ and $K$ be Schwartz functions with $f(0) = 1$ and $\hat{K}(0) = 0$, and set $f_N(x) = N^{-d}f(x/N), N \geq 1$. Then for any positive integer $\alpha \geq d + 1$ there exists a constant $C_\alpha$ such that

\[
|K * f_N(x)| \leq \left( \int_{|x-y| > \sqrt{N}} |K(x-y)| |f_N(y) - f_N(x)|dy \right. \\
\int_{|x-y| \leq \sqrt{N}} |K(x-y)| |f_N(y)|dy + C_\alpha N^{-d-1/2}(1 + |x/N|)^{-\alpha}
\]

(2.7) $\leq C_\alpha N^{-1/2}\left( \int_{\mathbb{R}^d} (1 + |x-y|)^{-\alpha}|f_N(y)|dy + N^{-d}(1 + |x/N|)^{-\alpha} \right)$.

This implies that

$$\lim_{N \to \infty} \frac{\|K * f_N\|_p}{\|f_N\|_p} = 0, \quad 1 \leq p < \infty.$$ 

Then the inclusion (2.2) holds by the following theorem.

**Theorem 2.2.** Let $B$ be a Banach space of tempered distributions with $S$ being dense in $B$. Assume that for any $\xi_0 \in \mathbb{R}^d$ there exists $\varphi_0 \in \mathcal{D}$ such that $\hat{\varphi}_0(0) = 1$ and

(2.8) $\lim_{N \to \infty} \frac{\|(m\hat{f}_{N, \xi_0})^\vee\|_B}{\|f_{N, \xi_0}\|_B} = 0$

for all Schwartz functions $m$ with $m(\xi_0) = 0$, where $\hat{f}_{N, \xi_0}(\xi) = \varphi_0(N(\xi - \xi_0))$.

If the Bochner-Riesz mean $B_\delta$ is bounded on $B$, then

(2.9) $\inf_{f \neq 0} \frac{\|(\lambda I - B_\delta)f\|_B}{\|f\|_B} = 0$ for all $\lambda \in [0, 1]$.

**Proof.** The infimum in (2.9) is obvious for $\lambda = 0$. So we assume that $\lambda \in (0, 1]$ from now on. Select $\xi_0 \in \mathbb{R}^d$ so that $(1 - |\xi_0|^2)_+^\delta = \lambda$. Then for sufficiently large $N \geq 1$,

(2.10) $$(\lambda I - B_\delta)f_{N, \xi_0} = (m_{\xi_0}f_{N, \xi_0})^\vee,$$

where $m_{\xi_0}(\xi) = (\lambda - (1 - |\xi|^2)_+^\delta)\psi(\xi - \xi_0)$ and $\psi \in \mathcal{D}$ is so chosen that $\psi(\xi) = 1$ for $|\xi| \leq (1 - |\xi_0|)/2$ and $\psi(\xi) = 0$ for $|\xi| \geq 1 - |\xi_0|$. Observe that
$m_{\xi_0} \in \mathcal{D}$ satisfies $m_{\xi_0}(\xi_0) = 0$. This together with (2.8) and (2.10) proves that
\[
\lim_{N \to \infty} \frac{\| (\lambda I - B_\delta) f_{N,\xi_0} \|_B}{\| f_{N,\xi_0} \|_B} = 0.
\]
Hence (2.9) is proved for $\lambda \in (0, 1]$.

**Remark 2.3.** For $\xi \in \mathbb{R}^d$, define modulation operator $M_\xi$ by
\[
M_\xi f(x) = e^{ix \xi} f(x).
\]
We say that a Banach space $B$ is *modulation-invariant* if for any $\xi \in \mathbb{R}^d$ there exists a positive constant $C_\xi$ such that
\[
\| M_\xi f \|_B \leq C_\xi \| f \|_B, \quad f \in B.
\]
Such a Banach space with modulation bound $C_\xi$ being dominated by a polynomial of $\xi$ was introduced in [Ta] to study oscillatory integrals and Bochner-Riesz means. Modulation-invariant Banach spaces include weighted $L^p$ spaces, Triebel-Lizorkin spaces $F_{p,q}^\alpha$, Besov spaces $B_{p,q}^\alpha$, Herz spaces $K_{p,q}^\alpha$, and modulation spaces $M_{p,q}^\alpha$, where $\alpha \in \mathbb{R}$ and $1 \leq p, q < \infty$ [G, Gr, LYH, T].

For functions $f_{N,\xi_0}, N \geq 1$, in Theorem 2.2,
\[
f_{N,\xi_0}(x) = e^{ix \xi_0(\varphi_0(N \cdot))}(x)
\]
and
\[
(mf_{N,\xi_0})^\vee(x) = e^{ix \xi_0(m_{\xi_0}(\varphi_0(N \cdot))^\vee(x)),
\]
where $m_{\xi_0}(\xi) = m(\xi + \xi_0)$ satisfies $m_{\xi_0}(0) = 0$. Then for a modulation-invariant Banach space $B$, the limit (2.8) holds for any $\xi_0 \in \mathbb{R}^d$ if and only if it is true for $\xi_0 = 0$. Therefore we obtain the following result from Theorem 2.2.

**Corollary 2.4.** Let $B$ be a modulation-invariant Banach space of tempered distributions with $\mathcal{S}$ being dense in $B$. Assume that there exists $\varphi_0 \in \mathcal{D}$ such that $\varphi_0(0) = 1$ and $\lim_{N \to \infty} \| (m \varphi_0(N \cdot))^\vee \|_B/\| \varphi_0(N \cdot) \|^\vee \|_B = 0$ for all Schwartz functions $m(\xi)$ with $m(0) = 0$. If the Bochner-Riesz mean $B_\delta$ is bounded on $B$, then its spectrum on $B$ contains the unit interval $[0, 1]$.

### 3 Remarks

In this section, we extend conclusions in Theorem 1.1 to weighted $L^p$ spaces, Triebel-Lizorkin spaces, Besov spaces, and Herz spaces.
3.1 Spectra on weighted $L^p$ spaces

Let $1 \leq p < \infty$ and $Q$ contain all cubes $Q \subset \mathbb{R}^d$. A positive function $w$ is said to be a Muckenhoupt $A_p$-weight if

$$\left(\frac{1}{|Q|} \int_Q w(x)dx\right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/p-1}dx\right)^{p-1} \leq C, \ Q \in Q$$

for $1 < p < \infty$, and

$$\frac{1}{|Q|} \int_Q w(x)dx \leq C \inf_{x \in Q} w(x), \ Q \in Q$$

for $p = 1$ [GF, M]. For $\delta > (d - 1)/2$, convolution kernel of the Bochner-Riesz mean $B_\delta$ is dominated by a multiple of $(1 + |x|)^{-\delta - (d+1)/2}$ and hence it is bounded on weighted $L^p$ space $L^p_w$ for all $1 \leq p < \infty$ and Muckenhoupt $A_p$-weights $w$. For $\delta = (d - 1)/2$, complex interpolation method was introduced in [SS] to establish $L^p_w$-boundedness of $B_\delta$ for all $1 < p < \infty$ and Muckenhoupt $A_p$-weights $w$. The reader may refer to [GF, LS, Lo] and references therein for $L^p_w$-boundedness of Bochner-Riesz means with various weights $w$. In this subsection, we consider spectra of Bochner-Riesz means on $L^p_w$.

**Theorem 3.1.** Let $\delta > 0, 1 \leq p < \infty$, and $w$ be a Muckenhoupt $A_p$-weight. If the Bochner-Riesz mean $B_\delta$ is bounded on $L^p_w$, then its spectrum on $L^p_w$ is the unit interval $[0, 1]$.

**Proof.** Denote the norm on $L^p_w$ by $\| \cdot \|_{p,w}$. By Theorems 2.1 and 2.2, and modulation-invariance of $L^p_w$, it suffices to prove

$$\|K * f\|_{p,w} \leq C\|K\|_{S_{d+1,0}} \|f\|_{p,w} \text{ for all } f \in L^p_w,$$

and

$$\lim_{N \to \infty} \|(m \varphi_0(N \cdot))^\vee\|_{p,w} = 0$$

for all Schwartz functions $\varphi_0$ and $m$ with $\hat{\varphi}_0(0) = 1$ and $m(0) = 0$.

Observe that $|K(x)| \leq C\|K\|_{S_{d+1,0}} (1 + |x|)^{-d-1}$. Then (3.1) follows from the standard argument for weighted norm inequalities [GF].

Recall that any $A_p$-weight is a doubling measure [GF]. This doubling property together with (2.7) leads to

$$\|(m \varphi_0(N \cdot))^\vee\|_{p,w}^p \leq CN^{-p/2} \|(\varphi_0(N \cdot))^\vee\|_{p,w}^p + CN^{-(d+1)/2}pw([-N, N]^d).$$
On the other hand, there exists $\epsilon_0 > 0$ such that $|\varphi_0'(x)| \geq |\varphi_0'(0)|/2 \neq 0$ for all $|x| \leq \epsilon_0$. This implies that

$$
(3.4) \quad \|(\varphi_0(N \cdot))'\|_{p,w}^p \geq C N^{-d p} w([-\epsilon_0 N, \epsilon_0 N]^d).
$$

Combining (3.3), (3.4) and the doubling property for the weight $w$, we establish the limit (3.2) and complete the proof. 

3.2 Spectra on Triebel-Lizorkin spaces and Besov spaces

Let $\phi_0$ and $\psi \in S$ be so chosen that $\hat{\phi}_0$ is supported in $\{\xi, |\xi| \leq 2\}$, $\hat{\psi}$ supported in $\{\xi, 1/2 \leq |\xi| \leq 2\}$, and

$$
\hat{\phi}_0(\xi) + \sum_{l=1}^{\infty} \hat{\psi}(2^{-l} \xi) = 1, \ \xi \in \mathbb{R}^d.
$$

For $\alpha \in \mathbb{R}$ and $1 \leq p, q < \infty$, let Triebel-Lizorkin space $F_{p,q}^\alpha$ contain all tempered distributions $f$ with

$$
\|f\|_{F_{p,q}^\alpha} := \|\phi_0 * f\|_p + \left(\sum_{l=1}^{\infty} 2^{l \alpha} \|\psi_l * f\|_q^q\right)^{1/q} < \infty,
$$

where $\psi_l = 2^{ld} \psi(2^l \cdot), l \geq 1$. Similarly, let Besov space $B_{p,q}^\alpha$ be the space of tempered distributions $f$ with

$$
\|f\|_{B_{p,q}^\alpha} := \|\phi_0 * f\|_p + \left(\sum_{l=1}^{\infty} 2^{l \alpha} \|\psi_l * f\|_p^q\right)^{1/q} < \infty.
$$

Next is our results about spectra of Bochner-Riesz means on Triebel-Lizorkin spaces and on Besov spaces.

**Theorem 3.2.** Let $\delta > 0$, $\alpha \in \mathbb{R}$ and $1 \leq p, q < \infty$. If the Bochner-Riesz mean $B_\delta$ is bounded on $F_{p,q}^\alpha$ (resp. $B_{p,q}^\alpha$), then its spectrum on $F_{p,q}^\alpha$ (resp. on $B_{p,q}^\alpha$) is the unit interval $[0,1]$.

**Proof.** For $\lambda \notin [0,1]$ and $\delta > 0$, both $B_\delta$ and $(\lambda I - B_\delta)^{-1} - \lambda^{-1} I$ are multiplier operators with compactly supported symbols. Therefore $B_\delta$ (resp. $(\lambda I - B_\delta)^{-1}$) is bounded on the Triebel-Lizorkin space $F_{p,q}^\alpha$ if and only if it is bounded on the Besov space $B_{p,q}^\alpha$ if and only if it is bounded on $L^p$. The above equivalence together with Theorem 1.1 yields our desired conclusions for spectra of Bochner-Riesz means on Triebel-Lizorkin spaces and on Besov spaces. 

\[\Box\]
3.3 Spectra on Herz spaces

For $\alpha \in \mathbb{R}$ and $1 \leq p, q < \infty$, let Herz space $K^{\alpha,q}_p$ contain all locally $p$-integrable functions $f$ with

$$\|f\|_{K^{\alpha,q}_p} := \|f\chi_{|\cdot|\leq 1}\|_p + \left( \sum_{l=1}^{\infty} 2^{l\alpha q} \|f\chi_{2^{l-1}<|\cdot|\leq 2^l}\|_p^q \right)^{1/q} < \infty,$$

where $\chi_E$ is the characteristic function on a set $E$. The boundedness of Bochner-Riesz means on Herz spaces is well studied, see for instance [LYH, Wa]. Following the argument used in the proof of Theorem 3.1, we have

**Theorem 3.3.** Let $\delta > 0, 1 \leq p, q < \infty$ and $\alpha > -d/p$. If the Bochner-Riesz mean $B_\delta$ is bounded on $K^{\alpha,q}_p$, then its spectrum on $K^{\alpha,q}_p$ is the unit interval $[0, 1]$.

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