Painlevé Analysis of Nonlinear Evolution Equations - An Algorithmic Method

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ABSTRACT

This paper develops an algorithmic method, also valid for multicomponent systems, for deriving the generalization of a Lax Pair directly from a general integrable nonlinear evolution equation via the use of truncated Painlevé expansions. Although it has been well-known for years that this method works for simple scalar integrable evolution equations, nevertheless no systematic procedure has been given that would work in general for scalar as well as for multicomponent systems. The method presented here systematizes the necessary operations in applying the Painlevé method to a general integrable evolution equation or system of equations. We demonstrate that by using only one singular manifold, and by following the concept of enforcing integrability at each step (referred to here as the Principle of Integrability), one is led, if the system is indeed integrable, to an appropriate generalization of a Lax Pair, although perhaps in nonlinear form, called a “Lax Complex”. One new feature of this procedure is that it utilizes, as needed, a technique from the well-known Estabrook-Wahlquist method, for determining necessary integrating factors. The end result of this procedure is to obtain a Lax Complex, whose integrability condition will contain the original evolution equation as a necessary condition. This in itself, is sufficient to ensure that the Lax Complex may be used to construct Bäcklund solutions of the evolution equation, to obtain Darboux Transformations, and also to obtain Hirota’s tau functions, in a manner analogous to the procedure for single component systems. The additional problem of finding a general procedure for the linearization of any Lax Complex is not treated in this paper. However, we do demonstrate that a particular technique, which can be derived self-consistently from the Painlevé-Bäcklund equations, has proven to be sufficient so far. The Nonlinear Schrödinger Equation is used to illustrate the method, and then the method is applied to obtain, for the first time, via the Painlevé method, a Lax Complex, using only one singular manifold, for the vector Manakov system.
1 Introduction

The techniques of Painlevé analysis [1] are by now well-known and widely employed in the area of testing nonlinear systems for integrability. Various additional developments which have taken place in this area over the past decade include investigations of the reasons underlying the technique’s success [2], the study of “higher-order” truncations [3], the so-called “Invariant” Painlevé analysis [4], the judicious application of two (or more) singular manifold functions where deemed necessary [5], and the use of truncated Painlevé expansions to obtain analytic solutions for both integrable and nonintegrable evolution equations (which we shall hereafter refer to as ”NLPDE”) [6]. We shall refer to select, relevant portions of these works subsequently.

Another branch of the subject, with a long history [7]-[10], involves the mutual interconnections among various features or properties of integrable systems. Such interconnections were considered from the perspective of Painlevé analysis in a seminal series of papers by Weiss [11]. These papers developed the approach, now known as ‘the singularity manifold method (SMM)’ of truncating the principal or general branch Painlevé singularity expansion for the solution of the system of NLPDE at the constant term. This imposes a specific choice of the singular manifold function and has come to called ‘the singular manifold’ (as opposed to the infinite expansion employed in the Painlevé test where this function is arbitrary). This singularity manifold function and the truncated ‘singular part’ expansion are then used to algorithmically derive an auto-Bäcklund Transformation (BT) between two different solutions of the NLPDE, and also to semi-algorithmically derive the associated linear scattering problem and the evolution operator, called the Lax Pair. Since the equations resulting from the use of this truncated expansion result in an auto-BT, they are often referred to as the ‘Painlevé-Bäcklund equations’, and this is a terminology we will employ for brevity and convenience.

Weiss’ original technique was extensively developed by others, notably in the encyclopaedic article on various aspects of Painlevé expansions by Newell and his collaborators [12] which, among numerous other things, extended the Weiss SMM to derive Hirota tau-functions as well. However, the original semi-algorithmic nature of the derivation of the Lax Pair in Weiss’ procedure persisted. On other words, a strictly algorithmic procedure did not exist. Some key point seemed to have not been detailed. One recent area of success in the Painlevé analysis of single-component integrable
NLPDE has been to extend the Weiss procedure significantly to algorithmically derive the so-called ‘Weiss substitution’ [11] which enables the Painlevé-Bäcklund equations to be linearized into a Lax Pair, thus removing the semi-algorithmic nature of the Weiss procedure [13, 14]. In addition, for single component systems, this technique can be employed to algorithmically derive various other features of the integrable system such as Miura Transformations, Darboux Transformations, multisoliton solutions, Hirota’s tau function, and similarity reductions (see [13, 14] for recent reviews). As is readily apparent, much of this work was motivated by earlier work in [10] and [12].

However, the above procedure works only sometimes for multicomponent systems and, where it does work, there is still a large element of plain guesswork or ad hoc manipulation involved in deriving the Lax Pair from the Painlevé-Bäcklund equations for such systems. In addition, some of these manipulations require ‘a priori’ information about the answer, or educated guesses about how the form of resulting nonlinear Ricatti equations can be linearized into the Lax Pair.

The purpose of the present work is to present a key point about the Weiss method, that seems to have been missed in earlier developments. We will demonstrate that the use of of this point will serve as an important guide in develop a procedure whereby the Lax Pair for integrable systems (either monocomponent or multicomponent) can be derived directly from the Painlevé-Bäcklund equations. Thus we are able to develop an entirely algorithmic and self-consistent method without requiring any ‘a priori’ or extraneous information about the system. This is not to say that the procedure for accomplishing this, even for a very simple and well-studied multicomponent system such as the Nonlinear Schrödinger Equation (NLS), is simple. The intermediate equations can become quite long and complicated. However, if one persists and continues to follow the guiding principle, then one not only is able to reduce the complexity of the resulting equations, but also is able to come out at the other end with a reduced set of Painlevé-Bäcklund equations, which we shall call the “Lax Complex”, since directly from it, upon linearization, one can obtain a Lax Pair, usually by a Weiss-type substitution. The algorithmic procedure for this latter linearization procedure is still to be worked out. Of course, once one has the linear Lax Pair (and one still has also the Painlevé-Bäcklund equations), the derivations of other properties of the system, such as Darboux Transformations, multisoliton solutions and so on, follows (see [14] for instance). An important point that we have found is that, with this algorithmic procedure, it will suffice to consider only a single singular manifold function. It will not be necessary to consider two or more singular manifold
functions (one for each branch of the Painlevé expansion) as done in some earlier studies [15] - [17]. More detailed comments on this aspect will be made subsequently at the appropriate places.

Before continuing, in order for us to better appreciate these points, and to better understand where we are leading to, let us digress a bit and consider the reverse process. First for some clarification of nomenclature. The NLPDE expresses the evolution of some field variable(s). These same field variable(s) will appear in the Lax Pair as potential(s). We will use these two different terms for the same quantity, depending on the context. When we speak of the NLPDE, we will usually refer to the “field variable(s)”. When we discuss the auto-Bäcklund transformation or a Lax Pair, we may refer to the field variables as the “potential(s)”. However, the field variable(s) and the potential(s) will always be the same quantity or object.

Now, the reverse process is given a Lax Pair, how does one obtain an auto-Bäcklund transformation? There is a well defined system for doing this [8]. Typically, one turns the Lax Pair into a nonlinear PDE system, by dividing the equations by one or another of the components of the eigenfunction(s) of the linear Lax pair, and then solving for the potential(s). Now, one can immediately see why a movable singular point would be present, because the component that one divides by, would have a movable zero. This follows since any such component typically has at least two linear independent solutions. Then by simply adjusting the linear combination of these linear independent solutions, one can place this zero anywhere he pleases. Thus given a NLPDE that has a Lax Pair, it naturally follows that the solutions of the NLPDE will have movable singularities.

Next, in order to understand what one has to do in order to go the opposite direction, let us summarize some key points about Lax Pairs and integrable NLPDE. First, what is the significance of a Lax Pair for a NLPDE? That significance can be simply stated to be: \textit{The integrability condition for this Lax Pair of the NLPDE, is the NLPDE.} Now, this is a very key point, and in fact, is the key reason for the success of the method of the Inverse Scattering Transform (IST). This significance has the consequences that, if by some means, we are able to construct solutions of that Lax Pair, then we have instantly, and unequivocally, constructed a solution of the NLPDE. The reason is simply that the NLPDE is the integrability condition for the Lax Pair. This is why, in the method of the IST, one never has to “solve” the nonlinear equation directly. Instead, one only has to solve the two linear problems in a Lax Pair, and one then automatically has a solution of the NLPDE, as a consequence of integrability.
Let us make a couple of further observations. One is that the integrability condition for a system of equations (such as a Lax Pair) is a key feature of these methods. Also, a Lax Pair does not have to be a linear system, as in the example of obtaining an auto-Bäcklund transformation. (We will later refer to such nonlinear Lax Pairs as "Lax Complexes".) Lastly, note that when we proceed from a given Lax Pair to the auto-Bäcklund transformation, that is done only after one has already had all integrability conditions of the Lax Pair completely satisfied. Consequently, if one wishes to reverse this direction of flow, and go from the auto-Bäcklund form to a Lax Pair, then one should use care and ensure that at each step of the procedure, the condition of integrability is maintained or obtained. From the above arguments, one sees that there is a principle here that needs to be carefully noted. In order to be definite, we will give this principle a name, and call it the “Principle of Integrability”. The statement of this “Principle of Integrability” should be the following: If an NLPDE (monocomponent or multicomponent) is integrable and if it has a Lax Pair, then the integrability conditions of the Painlevé-Bäcklund equations should themselves be sufficient to determine a Lax Complex. (After all, what else could be required?) Let us now define precisely what we mean by the term “Lax Complex”. We define a “Lax Complex” to be any system of equations, linear, quasilinear, or even nonlinear, whose integrability conditions contain the NLPDE as a condition. Thus a Lax Pair is also a Lax Complex, but not necessarily the reverse. A Lax Complex could contain a Lax Pair, and other auxiliary conditions as well. It need not be linear. The construction of the nonlinear system of equations from a Lax Pair, in the process of constructing an auto-Bäcklund transformation, is an example of a Lax Complex. The key point is that the integrability condition of the Lax Complex must contain the NLPDE. If one can nontrivially linearize a Lax Complex, then one obtains a Lax Pair, providing the integrability conditions for the Lax Pair is simply and only the NLPDE. 

Continuing, what we find to be workable is the addition of the Principle of Integrability to the Weiss method. This is not to say that to some degree, this not been realized before, because it indeed has been well known how important integrability were to these equations. Rather what we are saying is that we are simply emphasizing that one must ensure that integrability is continually

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1As a final comment, since the Lax Pair is so well known as a collection of linear equations, for which there exist standard and well known methods for proving existence of solutions and their analytic properties, it becomes quite imperative to reserve this name for such systems. Whence the need for the introduction of a term such as “Lax Complex”, that carries the connotation of containing the NLPDE as an integrability condition, but not necessarily being a linear system.
satisfied as we proceed through the Weiss procedure. Newly stated, the combination of the Principle of Integrability with the Weiss procedure can now be given as a set of invariant rules, which when precisely followed, will lead one from a NLPDE to a Lax Pair, if the NLPDE is integrable. These steps are: 1) Using the Weiss SMM, one obtains the Painlevé-Bäcklund equations. 2) One now must carefully proceed to ensure that all possible integrability conditions contained in these equations are satisfied. 3) Once this is accomplished, then one has what we shall call, a “reduced Painlevé-Bäcklund” set of equations. 4) This reduced set of equations contain the original NLPDE as a component. We now delete this original NLPDE from this set of equations. This will give us what we shall call the “tentative Lax Complex”. 5) We now check all integrability conditions of the tentative Lax Complex. If these integrability conditions contain the NLPDE, then the tentative Lax Complex becomes a fully fledged “Lax Complex”. 6) We now must find a linearization for the Lax Complex (or a method for generating solutions of the Lax Complex). Once a linearization is obtained, then this becomes a “Lax Pair”. In this paper, we mainly treat Steps 1)-5). A general algorithmic procedure for the linearization step will be the subject of a later paper. Let us note that we do not imply that one can always satisfy each step. If the NLPDE is not integrable, then at one or another step, one will fail. One could also fail by being unable to find a suitable solution for any step, due to the complexity of the resulting equations.

The rest of this paper is organized as follows. In Section 2, we illustrate the above algorithmic procedure by using the well-known Nonlinear Schrödinger Equation (NLS) as an example. In Section 3, the procedure will be applied to the Manakov vector NLS system. It will be seen that the procedure is quite general and goes through smoothly for even the Manakov system, where the derivation of the Lax Pair from the Painlevé-Bäcklund equations is otherwise highly nontrivial, even with the answer being known. To our knowledge, this is the first time that one has been able to obtain the Lax Pair for the Manakov vector NLS by the Painlevé method, using a single singular manifold function. Section 4 demonstrates a solution to the linearization problem of these nonlinear Lax Complexes, to the end of obtaining the usual linear scattering problems and evolution operators. In Section 5, we conclude by summarizing the results and commenting on possible further uses of the procedure developed here.
2 Algorithmic Procedure: The NLS Example

In this section, we shall apply the above basic algorithmic procedure to the well-known example of the Nonlinear Schrödinger Equation [1] - [8]. We shall initially follow Weiss’ original singularity manifold method (SMM) as treated in [11]. However, we shall see that even for this well-studied and now classical system, our approach will render completely algorithmic the two steps in the procedure which formerly required prior, extraneous information about the system. Note also that, contrary to the point made by earlier authors regarding the necessity of using two singular manifolds for NLS (see [15] - [17] for instance), we can indeed obtain all the requisite information for NLS by using only a singular manifold function.

The first step in our procedure will be to employ the standard Weiss SMM, i.e. to insert a Painlevé expansion truncated at the constant term and obtain the resulting Painlevé-Bäcklund equations (all of our algebra has been performed and cross-checked using both the symbolic computing languages, MATHEMATICA and MACSYMA). We shall then process these equations in a manner which will algorithmically and systematically lead to the Lax Complex. In the following calculations, we shall repeatedly refer to the treatment of the NLS in [12] using the Weiss SMM technique. This will serve to illustrate the new features of our procedure for this well-studied example, as well as what is achieved thereby.

We consider the NLS equation in the general form

\[
\begin{align*}
    u_t' &= \frac{i}{2}(u_{xx}' - 2(u')^2 v') \\
    v_t' &= -\frac{i}{2}(v_{xx}' - 2u'(v')^2).
\end{align*}
\]

(2.1)

Note that usually \( v' = \pm u'' \) for NLS, but we shall simply follow the usual Zakharov-Shabat/AKNS procedure and treat them as independent mathematical variables. Employing the Weiss SMM and plugging in the truncated principal branch Painlevé expansion [12] (with \( \phi \) the singular manifold function and the leading coefficients \( u_0 \) and \( v_0 \) arbitrary functions), we have

\[
\begin{align*}
    u' &= \frac{u_0}{\phi} + u \\
    v' &= \frac{v_0}{\phi} + v
\end{align*}
\]

(2.2)
which results in the following set of Painlevé-Bäcklund equations at the indicated orders in $\phi$:

\begin{align*}
0(\phi^{-3}) : \quad & u_0v_0 = \phi_x^2, \\
0(\phi^{-2}) : \quad & 2i\phi_t = -\left(\dot{\phi}_{xx} + 2\frac{u_0}{u_0} \phi_x\right) - 2u_0v - 4v_0u, \\
& -2i\phi_t = -\left(\dot{\phi}_{xx} + 2\frac{v_0}{v_0} \phi_x\right) - 2v_0u - 4u_0v, \\
(\phi^{-1}) : \quad & -2iu_{0t} = u_{0xx} - 2u^2v_0 - 4uv_0v, \\
& 2iv_{0t} = v_{0xx} - 2v^2u_0 - 4v_0uv, \\
(\phi^0) : \quad & u_t = \frac{i}{2}(u_{xx} - 2u^2v), \\
& v_t = -\frac{i}{2}(v_{xx} - 2v^2u).
\end{align*}

For comparison purposes, note that these are exactly the same equations following Eq. (4.16) on page 35 of [12]. In addition, note that (2.6) shows that, as usual for the Weiss truncation, the zero-order (or constant level) coefficients $u$ and $v$ in (2.2) satisfy the original NLS equation (2.1). Thus, once we know $u_0, v_0$ and $\phi$, the truncated expansion (2.2) represents an auto-BT between two solutions of the NLS. This is the completion of Step 1.

Next to Step 2, which is invariably the key step, and is the one that has not been systematically articulated and/or employed earlier. We want to determine if three functions, $u_0, v_0$ and $\phi$, satisfying these equations can exist. This requires systematically exhausting all integrability conditions on the singularity manifold function $\phi$ and the functions $u_0$ and $v_0$ which occur in the Painlevé-Bäcklund equations. It is clear that some conditions must be satisfied, since when we consider (2.3)-(2.5), we see that there are five equations determining three variables, upon allowing $u$ and $v$ to be determined by (2.6). Thus the system is overdetermined.

To this end, let’s systematically sort out what the conditions must be, for a consistent solution to exist. We take (2.6) to determine $u$ and $v$, while we can also take (2.5) to determine $u_0$ and $v_0$. These equations, by themselves, are internally consistent. Then we are left with the three equations, (2.3) and (2.4), all of which, some function $\phi$ must satisfy. To simplify matters a bit, we
solve Eq. (2.4) for $\phi_t$ and $\phi_{xx}$, obtaining

$$
\phi_{xx} = -(u_0 v + v_0 u)
$$

$$
\phi_t = -i \left[ \frac{u_0 v_0 x - v_0 u_0 x}{2\phi_x} + \frac{u_0 v - v_0 u}{2} \right]
$$

in which form, the necessary integrability conditions will be more readily be found.

From (2.3) and (2.7), there are three integrability conditions that must be satisfied. First, the differential of (2.3) with respect to $x$ must be consistent with (2.7a), second, the differential of the same with respect to $t$ must be consistent with (2.7b), and third, each equation in (2.7) must be consistent with respect to each other, requiring $(\phi_t)_{xx} = (\phi_{xx})_t$. The first condition gives us a new constraint

$$
\frac{u_{0x}}{u_0} + 2\frac{u\phi_x}{u_0} + \frac{v_{0x}}{v_0} + 2\frac{v\phi_x}{v_0} = 0. 
$$

This is a condition on $u_{0x}$ and $v_{0x}$. We find it convenient to separate (2.8) into individual expressions for $u_{0x}$ and $v_{0x}$, which would allow us to obtain further integrability conditions (on $u_0$ and $v_0$) easily. This separation is arbitrary and is not necessary. We do this by defining a new quantity, $\Gamma(x, t)$, as follows.

$$
\Gamma = \frac{u_{0x}}{u_0} + 2\frac{u\phi_x}{u_0}.
$$

Then, from (2.8) and (2.9), we have

$$
u_{0x} = \Gamma u_0 - 2u\phi_x, \tag{2.10a}$$

$$v_{0x} = -\Gamma v_0 - 2v\phi_x. \tag{2.10b}$$

where, at this stage, $\Gamma$ is totally arbitrary.

In previous treatments [12], one had assumed that $\Gamma_x$ would be zero. (Or perhaps it could have been that this point simply had not been clearly stated in that reference. Nevertheless, even though not stated there, this was indeed the only solution possible, as we shall now see.) Here we shall show clearly that that assumption was not necessary. This follows immediately from the next integrability condition on $\phi$. When we check the consistency of (2.3) and (2.7b) using the above
results for \( u_{0,x} \) and \( v_{0,x} \), we find
\[
\Gamma_x \phi_x u_0 v_0 = 0, \tag{2.11}
\]
whose only acceptable solution is
\[
\Gamma_x = 0. \tag{2.12}
\]
The last integrability condition on \( \phi \) is the internal consistency of (2.7). From all the above, it follows that
\[
(\phi_{xx})_t - (\phi_t)_{xx} = 0 \tag{2.13}
\]
and (2.7) is now internally consistent. That completes the integrability conditions on \( \phi \).

However, in the process, we have seen that additional necessary conditions, (2.10) and (2.12), have arisen. In particular, there will be integrability conditions between (2.5) and (2.10). Evaluating \( (u_{0x})_t - (u_{0t})_x = 0 \), we find another condition
\[
\Gamma_t = 0, \tag{2.14}
\]
and then evaluation of \( (v_{0x})_t - (v_{0t})_x = 0 \) yields nothing new. From (2.12) and (2.14), it follows that \( \Gamma \) must be a constant. Thus we replace \( \Gamma \) by \(-2i\zeta\) where \( \zeta \) is a constant. This completes Step 2.

For Step 3, we simply reproduce the Painlevé-Bäcklund equations, taking into account the results from Step 2. This gives
\[
\phi_x^2 = u_0 v_0, \tag{2.15a}
\]
\[
i\phi_t = \frac{1}{2}(v_0 u - u_0 v) + 2i\zeta \phi_x, \tag{2.15b}
\]
\[
\phi_{xx} = -(u_0 v + v_0 u), \tag{2.15c}
\]
\[
-2i u_{0t} = u_{0xx} - 2u^2 v_0 - 4u_0 uv, \tag{2.16a}
\]
\[
-2i v_{0t} = v_{0xx} - 2v^2 u_0 - 4v_0 uv, \tag{2.16b}
\]
\[
u_{0x} = -2i\zeta u_0 - 2u \phi_x, \tag{2.16c}
\]
\[
v_{0x} = 2i\zeta v_0 - 2v \phi_x, \tag{2.16d}
\]
\[ u_t = \frac{i}{2}(u_{xx} - 2u^2v), \quad (2.17a) \]
\[ v_t = -\frac{i}{2}(v_{xx} - 2v^2u) \quad (2.17b) \]

These equations are now internally consistent, and for definiteness, we will refer to these equations as the “reduced Painlevé-Bäcklund equations”. This completes Step 3.

Step 4 involves the simple action of deleting the original NLPDE from the reduced Painlevé-Bäcklund equations. This set of equations will be referred to as the “tentative Lax Complex”. Those equations are (2.15) and (2.16). That then completes Step 4.

Step 5 involves the additional action of checking the integrability conditions of the tentative Lax Complex, to ensure that its integrability condition is the NLPDE, and only the NLPDE. One may readily verify that this is the case, and thus Eqs. (2.15) and (2.16) are indeed a Lax Complex for the NLS. We are now in the position where, if we chose to and were able to, if we could find a solution of the Lax Complex, by whatever means, it would follow that \( u \) and \( v \) will satisfy (2.6). Therefore, instead of trying to directly solve (2.6), one may alternately solve (2.15) and (2.16). Of course, at the present moment, these latter equations appear to be even more complex than (2.6). Nevertheless, this completes Step 5.

The last step, Step 6, is the linearization of the Lax Complex whose general solution will not be treated here. As is well known [12], these particular equations (2.15)-(2.16) can be “linearized”, reducing them to the well known linear Lax Pair for (2.6), which has a method of solution by the IST.

So, let us briefly summarize the approach that we have used here. Take any evolution equation of interest. Using the Weiss SMM, determine the truncated Painlevé expansion. Collect the various orders of the Painlevé-Bäcklund expansion. Now impose all necessary integrability conditions on these Painlevé-Bäcklund equations. The only point that appears to require some ingenuity was the equivalent of (2.8). A large amount of algebra was actually eliminated by the introduction of the additional function \( \Gamma(x,t) \). Had we separated this by using a different form for \( \Gamma \) (say \( \tilde{\Gamma} = u_0 \Gamma \) instead), then (2.11) might not have occurred as obviously as it did. Nevertheless, with work, it still could have been obtained. (If nothing else, one could always borrow a trick from the Estabrook-Wahlquist method [18], whereby one assumes a most general form of \( \tilde{\Gamma}(\phi, u_0, v_0, \ldots) \), insert this into any condition determining \( \tilde{\Gamma} \), match coefficients and solve. Here we would have
obtained \( \tilde{T} = u_0 \cdot \text{constant} \). A nontrivial example of this will be found in the next section.) Once this analysis is complete, then the equations reduce to the reduced Painlevé-Bäcklund equations.

Now one makes the final test. Up to this point, one has been able to use the equivalent of (2.6) quite freely in evaluating all the integrability conditions. The question now is: Upon removing the NLPDE from the reduced Painlevé-Bäcklund equations, will the NLPDE still be an integrability condition? If so, then the reduced Painlevé-Bäcklund equations, minus the NLPDE, become a Lax Complex. Finding any solution or solutions of the Lax Complex will ensure that the nonlinear evolution equation will also be satisfied.

Of course, nonlinear equations are generally not easy to solve. So if possible, one would like to linearize the Lax Complex. However, that is a quite different (and additional) problem. It has nothing to do with the integrability of the original nonlinear evolution equation. It only concerns the method of solution for the Lax Complex.

We defer the linearization step to Section 4 where we shall illustrate one method for solving it, using a general procedure which will apply to the Lax Complex (2.15)–(2.16) of the NLS equation, as well as that for the Manakov vector NLS system. In order to illustrate the generality of the procedure we have developed in this section, we shall now address the Manakov vector NLS system.

3 The Manakov System

In this section, we shall illustrate the power of the algorithmic procedure developed in Section 2 further by applying it to the well-known vector NLS or Manakov equations. Note that there have been many earlier studies (see [19]-[21] and references therein) on this system, as well as generalizations to larger integrable hierarchies of NLS equations including uses of the Painlevé approach (see [19]-[21] and references therein). However, in the studies employing the Painlevé analysis, the Painlevé test is performed but then other approaches, such as Hirota’s Method, are employed to derive multisoliton solutions. Although the Lax Pair was obtained by algebraic techniques in [21] (and was of course also derived in Manakov’s original paper [22]), it has not been derived before from the Painlevé-Bäcklund equations. The advantage of deriving it from the Painlevé approach is of course that the formalism can then be employed to algorithmically obtain Darboux Transformations (and Miura Transformations to related equations, if applicable), multisoliton solutions, tau...
functions, and symmetries.

None of the existing singularity analysis methods, such as use of regular Painlevé expansions [13, 14], ‘invariant’ Painlevé expansions [16] - [19], or various Ricatti systems of Lax Complexes assumed ‘a priori’ [16, 23], succeeds in obtaining the Lax Pair of this system from the Painlevé-Backlund equations, even with the answer being known. The power of the algorithm developed in Section 2 should be judged in that light, especially as it leads to the Lax Pair in a completely logical fashion. Another relevant comment may shed further light on this matter. For a system such as the Manakov one (or larger systems), the Painlevé-Backlund equations become so complex that gleaning information from them by either algebraic dexterity alone or pure guesswork, begins to become virtually useless – many combinations and permutations of variables and manipulations exist, and finding the right combination, without an algorithm to follow, is like looking for a needle in a haystack. However, as we shall see here, following the Principle of Integrability leads us directly to the answer.

Consider the Manakov system

\[
\begin{align*}
    iq_1' t &= Dq_1'_{xx} + aq_1' (q_1' q_3' + q_2' q_4') \\
    iq_2' t &= Dq_2'_{xx} + aq_2' (q_1' q_3' + q_2' q_4') \\
    -iq_3' t &= Dq_3'_{xx} + aq_3' (q_1' q_3' + q_2' q_4') \\
    -iq_4' t &= Dq_4'_{xx} + aq_4' (q_1' q_3' + q_2' q_4').
\end{align*}
\] (3.1)

For the physical system, we usually have \( q_3' = \pm q_1'^* \), \( q_4' = \pm q_2'^* \), wherein \( q_3' \) and \( q_4' \) are related to the complex conjugates of \( q_1' \) and \( q_2' \). However, as usual in the Zakharov-Shabat/AKNS approach, and as we did earlier for NLS, we shall treat these quantities as independent variables. The usual leading-order dominant balance reveals that all four variables have poles of order one. Hence, we take the Weiss-type expansions, truncated at the constant level, for Step 1 of our procedure

\[
q_i' = \frac{q_{i0}}{\phi} + q_i, \quad i = 1, \ldots, 4.
\] (3.2)

Substituting (3.2) into (3.1), the leading order at \( \phi^{-3} \) gives that the leading order coefficients \( q_{i0}, i =
1, \ldots, 4, must satisfy the constraint

\[ q_{10}q_{30} + q_{20}q_{40} = -\frac{2D}{a} \phi_x^2, \] (3.3)

which is our first component of the Painlevé-Bäcklund equations. The other Painlevé-Bäcklund equations, which occur at orders \( O(\phi^{-2}) \) to \( O(1) \), are given in Appendix A. This completes Step 1.

Next, for Step 2, we must now systematically enforce all possible integrability conditions on these equations. Before we do this, we shall reorder the equations, so that we may more clearly see what are the necessary integrability conditions that need to be satisfied. Considering the equations in the Appendix A, we see that we may take (A5) through (A8) to define the functions \( q_{j0} \)'s, and then also may take (A9) through (A12) to define the functions \( q_j \)'s. Also, as usual, (A9)-(A12) show that the constant level coefficients \( q_i, i = 1, \ldots, 4 \) in (3.2) also satisfy the Manakov equations, so that (3.2) represents an auto-BT for this system.

Continuing, in addition to (3.3), this then leaves (A1) through (A4) to define the one function \( \phi \), for a total of five conditions on one function. As in Section 2 for NLS, two equations of (A1) through (A4) may be solved for the highest occurring \( t \) and \( x \) derivatives of \( \phi \). However, doing so and eliminating \( \phi_t \) and \( \phi_{xx} \) from the other two equations will, unlike for the NLS, provide two additional compatibility conditions. Thus we solve (A2) and (A4) simultaneously for the highest occurring spatial and temporal derivatives, which yields

\[
\phi_{xx} = \frac{-[-aq_{10}q_{20}q_{30}q_4 - 2aq_{10}q_{20}q_{30}q_{40} - aq_{10}q_2q_{30}q_{40} - 2a q_{10}q_{20}q_{30}q_{40} - 3aq_{20}^2q_{40}^2 - 3aq_{20}q_{40}^2 + 2D(q_{40}q_{20x} + q_{20}q_{40x})\phi_x]}{(2Dq_{20}q_{40})} \] (3.4a)

\[
\phi_t = \frac{-i[a_{10}q_{20}q_{30}q_4 - aq_{10}q_2q_{30}q_{40} + aq_{20}q_{40}]}{(2q_{20}q_{40})} \] (3.4b)

These equations are analogues of (2.7). Next, inserting these results into (A1) and (A3) and simplifying, yields the identities

\[
\frac{q_{10x} + q_1\phi_x}{q_{10}} = \frac{q_{20x} + q_2\phi_x}{q_{20}} \equiv \lambda, \] (3.5)
and

\[
\frac{q_{30,x} + q_3 \phi_x}{q_{30}} = \frac{q_{40,x} + q_4 \phi_x}{q_{40}} \equiv \chi. \tag{3.6}
\]

Now, in the above, we have arbitrarily introduced the two additional functions \(\lambda\) and \(\chi\). They are defined by these expressions. The first equality in each of these two equations, is the identity that follows from inserting (3.4a) and (3.4b) into either (A1) or (A3). As we did with \(\Gamma\) for the NLS, these arbitrary functions have been introduced so that we may now solve for the \(x\)-derivatives of the \(q_{j0}\)'s. The above two equations then become the analogues of (2.8) and (2.9) for the NLS. Solving explicitly for the derivatives, we then have

\[
q_{10,x} = \lambda q_{10} - q_1 \phi_x, \\
q_{20,x} = \lambda q_{20} - q_2 \phi_x, \\
q_{30,x} = \chi q_{30} - q_3 \phi_x, \\
q_{40,x} = \chi q_{40} - q_4 \phi_x. \tag{3.7}
\]

Next, we systematically enforce all integrability conditions on \(\phi\), and the \(q_{j0}\)'s. First, the compatibility of the constraint, (3.3), with \(\phi_{xx}\) may be imposed by using (3.4a) in the condition obtained by taking the \(x\) partial derivative of (3.3). After some computer algebra, this simplifies to

\[
\lambda + \chi = \frac{a}{2D\phi_x}(q_1 q_{30} + q_2 q_{10} + q_3 q_{40} + q_4 q_{20}). \tag{3.8}
\]

Next, as in the case of the NLS, we enforce the compatibility of (3.3) with the expression for \(\phi_t\). First we take the time partial of (3.3), employ (A5)-(A8) to eliminate the \(q_{j0t}\)'s, then use (3.4b) to eliminate every \(t\)-derivative of \(\phi\), and lastly, (3.7) to eliminate all the \(x\)-derivatives of \(q_{j0}\). Then also upon using (3.8) to eliminate \(\chi\), this yields the following differential equation for \(\lambda\):

\[
-4D^2 \phi_x^2 \partial_x \lambda - 2aD (q_2 q_{40} + q_1 q_{30}) \phi_x \lambda + 2aD (q_{10} q_{2,0} + q_{30} q_{1,0}) \phi_x \\
+ a^2 (q_{10} q_{30} + q_{20} q_{40}) (q_1 q_3 + q_2 q_4) = 0. \tag{3.9}
\]

For the reduction of this equation, we will find an old technique from the Estabrook-Wahlquist method [18] useful, which was alluded to, in the previous NLS section. Here, we can present a
nontrivial example of that technique. Let us assume that $\lambda$ can be expressed as a function of our basic variables. We take therefore, $\lambda = f(\phi_x, q_1, q_2, q_3, q_4, q_{10}, q_{20}, q_{30}, q_{40})$. Inserting this assumed form for $\lambda$ into (3.9), using the chain rule of differentiation, then using the expressions for $q_{j0,x}$ from (3.7) to eliminate those, and then eliminating any appearance of $\chi$ by means of (3.8), reveals two interesting facts. First, note that we have no constraints on $q_{j,x}$. Thus these quantities must be taken to be independent. From the resulting expression, it then follows that $\lambda$ must only be linear in $q_1$ and $q_2$, while being independent of $q_3$ and $q_4$. Furthermore, the explicit dependence on $q_1$ and $q_2$ must exactly be as:

$$\lambda = \frac{a}{2D\phi_x}(q_1q_{30} + q_2q_{40}) + \theta,$$

(3.10)

where the quantity $\theta$ includes all other possible dependence of $\lambda$ on $\phi_x$, $q_{10}$, $q_{20}$, $q_{30}$ and $q_{40}$. Inserting (3.10) into (3.9), reducing all $x$-derivatives as before, one obtains the very simple result of

$$\theta_x = 0 \quad \text{or} \quad \theta = \theta(t).$$

(3.11)

Next, we continue Step 2 by enforcing $(\phi_{xx})_t - (\phi_t)_{xx} = 0$. Taking the appropriate partial derivatives of (3.4a) and (3.4b) and using (A5)-(A8) and (3.8)/(3.9) to eliminate appropriate time and space derivatives of the $q_{j0}$’s, we find that due to (3.11), this condition is identically satisfied, as it was for the NLS.

Now, we have solutions for $\lambda$ and $\chi$ in terms of $\theta$, and we have all integrability conditions on $\phi$ satisfied, provided that the $x$-partials of $q_{j0}$ satisfy (3.7). However the latter must also be consistent with (A5)-(A8). Thus they will also have integrability conditions. Requiring now that $\partial_t q_{j0,x} = \partial_x q_{j0,t}$, for $j = 1, \ldots, 4$, we find only one further condition, and that is:

$$\partial_t \theta = 0, \text{thus } \theta = \text{constant} = 2i\zeta,$$

(3.12)

where $\zeta$ will be the spectral parameter.

As in Section 2, Step 3 involves identifying the “reduced Painlevé-Bäcklund” equations from the Painlevé-Bäcklund equations and the results of Step 2. As a consequence of the process of obtaining the integrability conditions, the form of several of these equations will simplify. Thus
we will now reproduce those equations here.

\[ 0 = q_{10} q_{30} + q_{20} q_{40} + \frac{2D}{a} \phi_x^2, \]

\[ \phi_{xx} = \frac{a}{2D} (q_{10} q_{30} + q_{20} q_{40} + q_{30} q_{10} + q_{40} q_{20}) , \]

\[ \phi_t = 4D \phi_x \zeta + \frac{ia}{2} (-q_{10} q_{30} - q_{20} q_{40} + q_{30} q_{10} + q_{40} q_{20}) \]  \hspace{1cm} (3.13)

\[ q_{10,x} = 2i \zeta q_{10} - q_1 \phi_x + \frac{aq_{10}}{2D \phi_x} (q_{10} q_{30} + q_{20} q_{40}) , \]

\[ q_{20,x} = 2i \zeta q_{20} - q_2 \phi_x + \frac{aq_{20}}{2D \phi_x} (q_{10} q_{30} + q_{20} q_{40}) , \]

\[ q_{30,x} = -2i \zeta q_{30} - q_3 \phi_x + \frac{aq_{30}}{2D \phi_x} (q_{30} q_{10} + q_{40} q_{20}) , \]

\[ q_{40,x} = -2i \zeta q_{40} - q_4 \phi_x + \frac{aq_{40}}{2D \phi_x} (q_{30} q_{10} + q_{40} q_{20}) \]  \hspace{1cm} (3.14)

\[ q_{10,t} = i \left\{ q_{10} \left[ 4D \zeta^2 - a \left( q_{10} q_{30} + \frac{1}{2} q_{20} q_{40} \right) \right] - \frac{a}{2} q_{10} q_{40} q_{20} \right\} + D \phi_x (i q_{10} - 2 q_1 \zeta) \]

\[ \frac{aq_{10}}{\phi_x} \left[ \zeta (q_{10} q_{30} + q_{20} q_{40} - i q_{20} q_{40} - i q_{30} q_{10}) \right] , \]

\[ q_{20,t} = i \left\{ q_{20} \left[ 4D \zeta^2 - a \left( q_{20} q_{40} + \frac{1}{2} q_{10} q_{30} \right) \right] - \frac{a}{2} q_{20} q_{30} q_{10} \right\} + D \phi_x (i q_{20} - 2 q_2 \zeta) \]

\[ \frac{aq_{20}}{\phi_x} \left[ \zeta (q_{10} q_{30} + q_{20} q_{40} - i q_{20} q_{40} - i q_{30} q_{10}) \right] , \]

\[ q_{30,t} = i \left\{ q_{30} \left[ -4D \zeta^2 - a \left( q_{30} q_{10} + \frac{1}{2} q_{20} q_{40} \right) \right] + \frac{a}{2} q_{20} q_{40} q_{30} \right\} + D \phi_x (-i q_{30} - 2 q_3 \zeta) \]

\[ \frac{aq_{30}}{\phi_x} \left[ \zeta (q_{10} q_{30} + q_{20} q_{40} + i q_{10} q_{30} + i q_{20} q_{40}) \right] , \]

\[ q_{40,t} = i \left\{ q_{40} \left[ -4D \zeta^2 - a \left( q_{40} q_{20} + \frac{1}{2} q_{10} q_{30} \right) \right] + \frac{a}{2} q_{10} q_{40} q_{30} \right\} + D \phi_x (-i q_{40} - 2 q_4 \zeta) \]

\[ \frac{aq_{40}}{\phi_x} \left[ \zeta (q_{10} q_{30} + q_{20} q_{40} + i q_{20} q_{40} + i q_{10} q_{30}) \right] . \]  \hspace{1cm} (3.15)

When the above equations are combined with the Manakov equations, (A9) - (A12), we have the reduced Painlevé-Bäcklund Equations. That then completes Step 3. For Step 4, one is to identify the “tentative Lax Complex”, which is the reduced Painlevé-Bäcklund equations minus the NLPDE. That would simply be the equations given above, (3.13) - (3.15).

Step 5 of our procedure consists of checking the integrability conditions of the “tentative Lax Complex”, to ensure that these still yield the governing NLPDE, and only the NLPDE. One may readily verify that such is so. Thus these equations are also a full Lax Complex.

Note that once again, we are led to this logically and directly, and with no prior or extraneous
knowledge regarding the Lax pair, by starting from the Painlevé-Bäcklund equations and following the procedure of Section 2. The comments made in the second paragraph of this section on the non-trivial nature of deriving the Lax Pair from the Painlevé formulation, without following this procedure, need to be highlighted at this point, and they should serve to clearly highlight the efficacy and power of this procedure. One method for the linearization of this nonlinear Lax Complex will be considered in Section 4.

4 Linearization into the Linear Scattering Problem

In this section, we shall linearize the nonlinear Lax Complexes obtained algorithmically in Sections 2 and 3 into their usual linear forms.

The strategy for doing this will again be algorithmic. It involves performing a leading-order analysis of the equations obtained earlier from the Painlevé analysis and we next consider this for the NLS equations in Section 2.

First, we consider a step which has repeatedly been used [9,11] in obtaining the Lax Pair of single component (or simpler multicomponent) systems from the Painlevé-Bäcklund equations. Defining the standard quantities

\begin{align}
V &\equiv \frac{\phi_{xx}}{\phi_x} \\
C &\equiv \frac{\phi_t}{\phi_x}
\end{align}

their cross-derivative condition \( \phi_{xxt} = \phi_{txx} \) yields

\[ V_t = (C_x + CV)_x \] (4.2)

Considering this as a nonlinear PDE connecting \( C \) and \( V \), we first perform a leading-order analysis (or the first step of the SMM) on these equations. Assuming leading behaviors

\begin{align}
V &\sim V_0 \psi^a \\
C &\sim C_0 \psi^b
\end{align}

(4.3)

(4.4)
and balancing the most singular terms (those within braces) in (4.2) yields

\[ a = -1 \quad \text{and} \quad V_0 \sim -b \psi_x \]  

(4.5)

Next, we return to the equations obtained from the Painlevé analysis in Section 2 and perform a leading order analysis on them. Using (2.3) we may write

\[ u_0 = A(x,t) \phi_x \]  

(4.6a)

\[ v_0 = \phi_x / A(x,t) \]  

(4.6b)

for some arbitrary function \( A(x,t) \). Using (2.11), we then have

\[ u \sim -\frac{A}{2} \left[ \frac{A_x}{A} + V \right] \]  

(4.7a)

\[ v \sim -\frac{1}{2A} \left[ -\frac{A_x}{A} + V \right]. \]  

(4.7b)

Using (4.5) in (4.3), and assuming a leading-order behavior

\[ A(x,t) \sim \psi^k \]  

(4.8)

the most singular terms in (2.6) yield (at order \( \psi^{k-3} \))

\[ [2(k - 1)(k - 2) - (k^2 - b^2)](k - b) = 0 \]  

(4.9)

and

\[ [2(k + 1)(k + 2) - (k^2 - b^2)](k + b) = 0. \]  

(4.10)

Clearly, \( k = b \) and \( k = -b \) satisfy (4.9) and (4.10) respectively. For these two equations to be satisfied simultaneously,

a. either, there must be a value of \( k \) different from these two values and satisfying both equations

b. or, there is (are) a value (values) of \( k \) satisfying both (4.9) and (4.10). It is easy to check that \( k = b \) AND \( k = -b \) satisfy both equations for

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Using the first value $b = -1$, the corresponding $k$ values are $k = b = -1$ and $k = -b = 1$. Thus, there are two solutions or branches, and, as usual [9,11], we take distinct singularity manifolds $\psi_1$ and $\psi_2$ for them. Thus, using (4.8) and the two $\psi$’s for the two values of $k$ yields

$$\frac{A_x}{A} = \frac{k_1\psi_{1x}}{\psi_1} + \frac{k_2\psi_{2x}}{\psi_2} = \frac{\psi_{1x}}{\psi_1} - \frac{\psi_{2x}}{\psi_2}$$

or

$$A = (\psi_1/\psi_2).$$

Using (4.5) with $b = -1$ and the two $\psi$’s, (4.3) implies

$$V \equiv \frac{\phi_{xx}}{\phi_x} = \frac{\psi_{1x}}{\psi_1} + \frac{\psi_{2x}}{\psi_2}$$

or

$$\phi_x = \psi_1\psi_2.$$ 

Thus, from (4.6), we obtain the usual linearizing transformations (or AKNS squared eigenfunction substitutions)

$$u_0 = \psi_1^2, \quad v_0 = \psi_2^2.$$

Note that these have now been obtained self-consistently from a leading-order analysis of the Painlevé-Bäcklund equations.

As is very well-known (see [12] for instance), (4.15)/(4.16) linearize the nonlinear Lax Pair (2.5) and (2.12) into the temporal and spatial (eigenvalue) parts of the linear Lax Pair or scattering problem.

The case $b = -2$ would similarly yield the linearizing transformations

$$\phi_x = \tilde{\psi}_1^2\tilde{\psi}_2^2$$

$$u_0 = \tilde{\psi}_1^4$$

$$v_0 = \tilde{\psi}_2^4.$$
It is trivially obvious that (4.17) is just the earlier linearizing transformation (4.15)/(4.16) under the identification \( \tilde{\psi}_1^2 \equiv \psi_1, \tilde{\psi}_2^2 = \psi_2 \).

An analogous, but somewhat lengthier, leading-order analysis of the Painlevé-Backlund equations given in Section 3 for the Manakov system yields the linearizing substitutions

\[
\phi_x = c_1 \psi_1 \psi_2 \quad (4.18)
\]

\[
q_{10} = c_2 \psi_1^2 \quad (4.19a)
\]

\[
q_{20} = - \left( \frac{2D}{a} c_1^2 + c_2 c_3 \right) \psi_1^2 \psi_2 / c_4 \psi_3 \quad (4.19b)
\]

\[
q_{30} = c_3 \psi_2^2 \quad (4.19c)
\]

\[
q_{40} = c_4 \psi_2 \psi_3 \quad (4.19d)
\]

where

\[
\frac{c_2}{c_1} = - \frac{2D}{(4D-a)} \sqrt{\frac{a}{2D}} \quad (4.20a)
\]

\[
\frac{c_3}{c_1} = \frac{c_4}{c_1} = \sqrt{\frac{2D}{a}}. \quad (4.20b)
\]

Using (4.18)-(4.20), the nonlinear Lax Pair given for the Manakov system in Section 3 is, after some computation, linearized to the usual linear scattering problem [22]

\[
\psi_{1x} = \frac{1}{\sqrt{2D}} \left[ -i \lambda \psi_1 + q_1 \sqrt{a} \psi_2 + q_2 \sqrt{a} \psi_3 \right]
\]

\[
\psi_{2x} = \frac{1}{\sqrt{2D}} \left[ -q_3 \sqrt{a} \psi_1 + i \lambda \psi_2 \right] \quad (4.21)
\]

\[
\psi_{3x} = \frac{1}{\sqrt{2D}} \left[ -q_4 \sqrt{a} \psi_1 + i \lambda \psi_3 \right]
\]
and the time evolution equations

\[
\begin{align*}
\psi_{1t} & = A\psi_1 + B\psi_2 + C\psi_3 \\
\psi_{2t} & = D\psi_1 + E\psi_2 + F\psi_3 \\
\psi_{3t} & = G\psi_1 + H\psi_2 + I\psi_3
\end{align*}
\]

\[
\begin{align*}
A &= \frac{ia}{2}(|q_1|^2 + |q_2|^2), & B &= \lambda \sqrt{a} q_1 + \frac{i\sqrt{2aD}}{2} q_{1t} \\
C &= \lambda \sqrt{a} q_1 + \frac{i\sqrt{2aD}}{2} q_{2t} \\
D &= -\lambda \sqrt{a} q_3 + \frac{i\sqrt{2aD}}{2} q_{3t} \\
E &= 2i\lambda^2 - \frac{ia}{2} |q_1|^2 \\
F &= -\frac{ia}{2} q_2 q_3 \\
G &= -\lambda \sqrt{a} q_4 + \frac{i\sqrt{2aD}}{2} q_{4t} \\
H &= -\frac{ia}{2} q_1 q_4, & I &= 2i\lambda^2 - \frac{ia}{2} |q_2|^2
\end{align*}
\]

whose compatibility yields the original Manakov system (3.1).

5 Summary and Prospects

In summary, we have developed a procedure for multicomponent integrable systems which may be employed to algorithmically and directly obtain the Lax Pair from the Painlevé-Bäcklund equations resulting from using the usual Weiss singular manifold method. Of course, once the Lax Pair is obtained, various other properties of the system such as Darboux Transformations, Miura Transformations to related systems (if any), multisoliton solutions, tau functions and so on may be systematically derived in a manner analogous to single component systems [13, 14].

As mentioned in the context of the Manakov model, the efficacy and directness of this procedure becomes apparent when compared to earlier techniques. Of the earlier techniques, the use of the usual Weiss SMM using regular Painlevé expansions truncated at the constant level (reviewed in [13] and [14]) fails for all but the simplest multicomponent systems. The reason is that the number of possible algebraic permutations and combinations is so large that finding the correct one becomes prohibitively complex. On the other hand, the approach using the invariant technique, or extensions
thereof [15] - [17], [23] all rely upon a pre-assumed form of nonlinear Lax Pair in projective Ricatti form. Once again, these techniques fail for all but the simplest multicomponent systems because of the difficulty of guessing the correct form of Ricatti system. An underlying reason for the failure of these approaches may quite well be that the Lax Pairs of such complicated multicomponent systems may themselves not have the usual Zakharov-Shabat/AKNS forms.

In conclusion, we expect that this technique will be of significant value in future work. This is likely to be particularly true for investigations of various new integrable multicomponent systems, such as recently derived integrable hierarchies, or various systems of interest in different application areas.

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References


APPENDIX A

The Painlevé-Bäcklund equations for the Manakov system at various orders in $\phi$ are:

$O(\phi^{-2})$:

\[
aq_{10}^2q_3 + 2aq_{10}q_{10}q_{30} + aq_{10}q_{20}q_4 + aq_{10}q_{20}q_{40} + aq_{10}q_{20}q_4 + iq_{10}\phi_t - 2D\phi_x q_{10x} - Dq_{10}\phi_{xx} = 0 \tag{A1}
\]

\[
aq_{10}q_{20}q_{30} + aq_{10}(q_{20}q_3 + q_{20}q_{30}) + aq_{20}^2q_4 + 2aq_{20}q_{20}q_4 + iq_{20}\phi_t - 2D\phi_x q_{20x} - Dq_{20}\phi_{xx} = 0 \tag{A2}
\]

\[
2aq_{10}q_{30}q_3 + aq_{10}^2q_{30} + aq_{20}q_{30}q_4 + aq_{20}q_{30}q_{40} + aq_{20}q_{30}q_{40} - iq_{30}\phi_t - 2D\phi_x q_{30x} - Dq_{30}\phi_{xx} = 0 \tag{A3}
\]

\[
aq_{10}q_{30}q_{40} + 2aq_{20}q_{40}q_4 + aq_{20}q_{40} + aq_{10}(q_{30}q_4 + q_{30}q_{40}) - iq_{40}\phi_t - 2D\phi_x q_{40x} - Dq_{40}\phi_{xx} = 0 \tag{A4}
\]

$O(\phi^{-1})$:

\[
aq_{10}^2q_{30} + aq_{10}q_{20}q_4 + aq_1(2q_{10}q_3 + q_{20}q_4 + q_{20}q_{40}) - iq_{10t} + Dq_{10xx} = 0 \tag{A5}
\]

\[
aq_{10}q_{20}q_3 + aq_1(q_{20}q_3 + q_{20}q_{30}) + 2aq_{20}q_{20}q_4 + aq_{20}^2q_4 - iq_{20t} + Dq_{20xx} = 0 \tag{A6}
\]

\[
aq_{10}q_{20}^2q_3 + 2aq_{10}q_{30}q_4 + aq_{20}q_{30}q_4 + aq_{20}q_{30}q_{40} + iq_{30t} + Dq_{30xx} = 0 \tag{A7}
\]

\[
aq_{10}q_{30}q_4 + aq_{20}q_{40}^2 + 2aq_{20}q_{40}q_4 + aq_1(q_{30}q_4 + q_{30}q_{40}) + iq_{40t} + Dq_{40xx} = 0 \tag{A8}
\]

$O(1)$:

\[
iq_{1t} = Dq_{1xx} + aq_1(q_{1}q_3 + q_{2}q_4) \tag{A9}
\]

\[
iq_{2t} = Dq_{2xx} + aq_2(q_{1}q_3 + q_{2}q_4) \tag{A10}
\]

\[
-iq_{3t} = Dq_{3xx} + aq_3(q_{1}q_3 + q_{2}q_4) \tag{A11}
\]

\[
-iq_{4t} = Dq_{4xx} + aq_4(q_{1}q_3 + q_{2}q_4) \tag{A12}
\]