Structure Preserving Methods for Damped Hamiltonian PDEs

Brian E. Moore, Laura Noreña, and Constance Schober

brian.moore@ucf.edu

University of Central Florida

International Workshop on Nonlinear and Modern Mathematical Physics,
University of South Florida, Tampa, March 10, 2013

Thanks to the National Science Foundation for partial funding
These equations can be written

\[ K z_t + L z_x = \nabla_z S(z) \]

- \( K \) and \( L \) are skew-symmetric matrices
- \( z = z(x, t) \) is the vector of state variables
- \( S \) is smooth

Examples include KdV, Boussinesq, Zakharov-Kuznetsov, nonlinear Schrödinger, nonlinear wave equations, etc.
**General Conformal Conservation Laws**

\[ \mathbf{K}_z\frac{\partial}{\partial t} + \mathbf{L}_z\frac{\partial}{\partial x} = \nabla_z S(z) \] has a conservation law \( \partial_t P + \partial_x Q = 0 \), implies

\[ \mathbf{K}_z\frac{\partial}{\partial t} + \mathbf{L}_z\frac{\partial}{\partial x} = \nabla_z S(z) - \frac{a}{2} \mathbf{K}_z - \frac{b}{2} \mathbf{L}_z \]

has a conformal conservation law if it satisfies

\[ \partial_t P + \partial_x Q = -aP - bQ \]

where \( a \) and \( b \) are positive real constants.


Examples of Conservation Laws for
\[ Kz_t + Lz_x = \nabla_z S(z) \]

- Energy: inner product of the equation with \( z_t \)
  \[ \partial_t \left( S(z) + \frac{1}{2}(z^T L z) \right) + \partial_x \frac{1}{2}(z^T L z_t) = 0 \]

- Momentum: inner product of the equation with \( z_x \)
  \[ \partial_x \left( S(z) + \frac{1}{2}(z_t^T K z) \right) + \partial_t \frac{1}{2}(z^T K z_x) = 0 \]

- Linear Symmetries: inner product with \( B z \)
  \[ \partial_t (z^T K B z) + \partial_x (z^T L B z) = 0 \]

- Multi-Symplectic: if \( U \) and \( V \) satisfy the variational equation
  \[ \partial_t \langle K U, V \rangle + \partial_x \langle L U, V \rangle = 0 \]
Examples of Conformal Conservation Laws for

\[ Kz_t + Lz_x = \nabla_z S(z) - \frac{a}{2}Kz - \frac{b}{2}Lz \]

- **Energy**: inner product of the equation with \( z_t \)

\[ \partial_t (S(z) + \frac{1}{2}(z_x^T L z)) + \partial_x \frac{1}{2}(z^T L z_t) = -\frac{b}{2}(z^T L z_t) \]

- **Momentum**: inner product of the equation with \( z_x \)

\[ \partial_x (S(z) + \frac{1}{2}(z_t^T K z)) + \partial_t \frac{1}{2}(z^T K z_x) = -\frac{a}{2}(z^T K z_x) \]

- **Linear Symmetries**: inner product with \( Bz \)

\[ \partial_t (z^T KB z) + \partial_x (z^T LB z) = -a(z^T KB z) - b(z^T LB z) \]

- **Multi-Symplectic**: if \( U \) and \( V \) satisfy the variational equation

\[ \partial_t \langle KU, V \rangle + \partial_x \langle LU, V \rangle = -a \langle KU, V \rangle - b \langle LU, V \rangle \]
Numerical Preservation

Suppose we have the following PDE and Conformal CL

\[ Kz_t + Lz_x = \nabla S(z) - \frac{a}{2}Kz \]

\[ \partial_t P + \partial_x Q = -aP \]
Numerical Preservation

Suppose we have the following PDE and Conformal CL

\[ Kz_t + Lz_x = \nabla S(z) - \frac{a}{2}Kz \quad \Rightarrow \quad \partial_t \hat{P} + \partial_x Q = -aP \]

Integrating with appropriate boundary conditions, yields

\[ \partial_t \hat{P} = -a \hat{P} \quad \iff \quad \hat{P}(t) = \exp(-at)\hat{P}(0) \quad \text{with} \quad \hat{P} = \int P \, dx. \]
Numerical Preservation

Suppose we have the following PDE and Conformal CL

\[ K z_t + L z_x = \nabla S(z) - \frac{a}{2} K z \]

\[ \partial_t P + \partial_x Q = -aP \]

Integrating with appropriate boundary conditions, yields

\[ \partial_t \hat{P} = -a\hat{P} \quad \iff \quad \hat{P}(t) = \exp(-at)\hat{P}(0) \quad \text{with} \quad \hat{P} = \int P \, dx. \]

A numerical method preserves this property if it satisfies

\[ \hat{P}^{i+1} = \exp(-a\Delta t)\hat{P}^i \quad \text{with} \quad \hat{P}^i = \sum_n P^{n,i} \Delta x. \]
Numerical Preservation

Suppose we have the following PDE and Conformal CL

\[
K_z t + L z_x = \nabla S(z) - \frac{a}{2}Kz \quad \partial_t P + \partial_x Q = -aP
\]

Integrating with appropriate boundary conditions, yields

\[
\partial_t \hat{P} = -a\hat{P} \quad \iff \quad \hat{P}(t) = \exp(-at)\hat{P}(0) \quad \text{with} \quad \hat{P} = \int P \, dx.
\]

A numerical method preserves this property if it satisfies

\[
\hat{P}^{i+1} = \exp(-a\Delta t)\hat{P}^i \quad \text{with} \quad \hat{P}^i = \sum_n P^{n,i} \Delta x.
\]

Note: \( \partial_t P < 0 \) is typically considered preserved if \( P^{i+1} < P^i \)
Splitting Methods, McLachlan and Quispel (2002)

\[ z_t = F_H(z) + F_D(z) \]

- Solve \( z_t = f_H(z) \) with conservative method \( z^{i+1} = \Psi_{\Delta t}(z^i) \).
- Solve \( z_t = Dz \) exactly with flow map \( \Phi_t(z) = \exp(Dt)z \).
- Original system is solved by composing these flow maps.
Splitting Methods, McLachlan and Quispel (2002)

$$z_t = F_H(z) + F_D(z)$$

1. Solve $z_t = f_H(z)$ with conservative method $z^{i+1} = \Psi_{\Delta t}(z^i)$.
2. Solve $z_t = Dz$ exactly with flow map $\Phi_t(z) = \exp(Dt)z$.
3. Original system is solved by composing these flow maps.

**Example**: Define non-standard difference/average operators

$$D^a_t z = \frac{z^{i+1} - e^{-a\Delta t}z^i}{\Delta t} \quad \text{and} \quad A^a_t z = \frac{z^{i+1} + e^{-a\Delta t}z^i}{2}$$

Implicit midpoint for $z_t = F_H(z)$ is $D^0_t z = F_H (A^0_t z)$.

Splitting method for $z_t = F_H(z) + F_D(z)$ is $D^{a/2}_t z = F_H \left( A^{a/2}_t z \right)$.
Structure-Preservation

A non-standard 1-step FD method preserves a conformal CL, if it has a discrete product rule and the corresponding CL is preserved by the conservative method upon which it is based.

Example: The conformal box scheme

\[
K \left( D_{t}^{a/2} A_{x}^{b/2} z \right) + L \left( D_{x}^{b/2} A_{t}^{a/2} z \right) = \nabla S \left( A_{t}^{a/2} A_{x}^{b/2} z \right)
\]

satisfies

\[
D_{t}^{a} \left< A_{x}^{b/2} z, KB A_{x}^{b/2} z \right> + D_{x}^{b} \left< A_{t}^{a/2} z, LB A_{t}^{a/2} z \right> = 0.
\]

Sum over spatial index with \( b = 0 \) and appropriate BCs gives

\[
\hat{P}^{i+1} = \exp(-a \Delta t) \hat{P}^{i} \quad \text{with} \quad \hat{P}^{i} = \sum_{n} \left< A_{x}^{0} z, KB A_{x}^{0} z \right> \Delta x.
\]
Nonlinear Schrödinger with Dissipation

\[ i\psi_t + \psi_{xx} + V'(|\psi|^2)\psi + i\frac{a}{2}\psi = 0 \]

Can be written \(Kz_t + Lz_x = \nabla_z S(z) - \frac{a}{2}Kz\) with

\[
J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} -J & 0 \\ 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix},
\]

\[ z = [v, w, p, q]^T, \quad S(z) = \frac{1}{2}(p^2 + q^2 + V(v^2 + w^2)) \]

Plane wave solution for \(V'(|\psi|^2) = \gamma|\psi|^2\) is

\[ \psi_0(t) = Ae^{-at/2} \exp \left( i\gamma|A|^2 \left( \frac{1 - e^{-at}}{a} \right) \right) \]
Comparison of Methods for Plane Wave

\[ a = \Delta t = 0.01, \quad L = 2\pi, \quad N = 64, \quad A = 1/2, \quad \gamma = 2 \]
Properties to Preserve

- **Conformal Momentum Conservation Law**

\[
\partial_t (vq - wp) + \partial_x \left( p^2 + q^2 + V(v^2 + w^2) + wv_t - vw_t \right) = -a(vq - wp),
\]

and integrating with respect to \( x \) gives

\[
M(t) = e^{-at} M(0) \quad \text{with} \quad M(0) = \int (vq - wp) \, dx.
\]

- **Conformal Norm Conservation Law**

\[
\partial_t (v^2 + w^2) + 2 \partial_x (vw_x - wv_x) = -a(v^2 + w^2)
\]

and integrating with respect to \( x \) gives

\[
N(t) = e^{-at} N(0) \quad \text{with} \quad N(0) = \int (v^2 + w^2) \, dx.
\]
Comparison of Methods for Plane Wave

Left: $N(t)$, Right: $M(t)$
Comparison of Methods for Plane Wave

\[ \mathcal{N}^{i+1} = e^{-a \Delta t} \mathcal{N}^i \implies a - \frac{1}{\Delta t} \ln \left( \frac{\mathcal{N}^i}{\mathcal{N}^{i+1}} \right) = 0 \]
Residual in Local Conformal Conservation Law

\[ a = \Delta t = 0.01, \quad N = 64, \quad T = 10, \quad u_0 = 0.5(1 + 0.1 \cos \mu x + 0.4 \cos 2\mu(x - L/4)) \]